

Operations Research

Lecture 6

How to be efficient?

Dual Simplex Method

Dual Simplex Method *condition*

Case1: If it is maximize then all the cost coefficient must be negative

Example:

Maximize: $Z = -2x_1 - 5x_2 - 12x_3$ so the **super optimal** solution for Z is Zero
which is satisfied from the initial basic solution where
 $x_1 = x_2 = x_3 = 0$

OR

Case2: If it is Minimize then all the cost coefficient must be Positive

Minimize: $Z = 2x_1 + 5x_2 + 12x_3$ so the **super optimal** solution for Z is Zero
which is satisfied from the initial basic solution where
 $x_1 = x_2 = x_3 = 0$

Dual Simplex Method Steps

1. Turn all constraints into less than or equal constraints.
2. Determine the output variable which has the most negative number in RHS.
3. Determine the input variable by calculating the ratio between the last row and the magnitude of negative numbers only in the pivot row, then consider the input variable is the one that has the smallest ratio.
4. Apply the same rules as in the regular simplex method until RHS has no negative numbers.

Example 1:

If we have the following LP problem:

$$\text{Max } Z = -4x_1 - 6x_2 - 18x_3$$

Subject to:

$$x_1 + 3x_3 \geq 3$$

$$2x_2 + 2x_3 \geq 5$$

$$x_1, x_2, x_3 \geq 0$$

Find:

The optimal solution of Primal Problem using the dual simplex method.

Example 1 cont...

$$\text{Max } Z = -4x_1 - 6x_2 - 18x_3$$

Subject to:

$$x_1 + 3x_3 \geq 3$$

$$2x_2 + 2x_3 \geq 5$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

(1) The optimal solution using the Primal simplex method.

Step 1:

Inequalities transfer to (\leq) formula as follows:

$$\text{Max } Z = -4x_1 - 6x_2 - 18x_3$$

Subject to:

$$-x_1 - 3x_3 \leq -3$$

$$-2x_2 - 2x_3 \leq -5$$

$$x_1, x_2, x_3 \geq 0$$

Step 2 :

The standard formula as follows:

$$\text{Max } Z = -4x_1 - 6x_2 - 18x_3 + 0S_1 + 0S_2$$

Subject to:

$$-x_1 - 3x_3 + S_1 = -3$$

$$-x_2 - 2x_3 + S_2 = -5$$

$$x_1, x_2, x_3, S_1, S_2 \geq 0$$

Step 3: Make a table simplex:

First stage:

Entering Variable

basic variable	x_1	$x_2 \downarrow$	x_3	S_1	S_2	b_i
Z_j	-4	-6	-18	0	0	0
S_1 0	-1	0	-3	1	0	-3
S_2 0	0	*-1	-2	0	1	-5
	4	6	18	0	0	
		6/1=6	18/2=9			

Leaving Variable



Most negative

Min ratio

Second stage: Entering Variable

Leaving Variable

basic variable	x_1	x_2	$x_3 \downarrow$	S_1	S_2	b_i
C_i	-6	-4	-18	0	0	
$\leftarrow 0 \ S_1$	-1	0	-3*	1	0	-3
-4 x_2	0	1	2	0	-1	5
	6	0	10	0	4	
Ratio	6/1		10/3			

Most negative

Min Ratio

$R1 * -1/3$

$R2 + 2/3 R1$

Third stage: The optimal solution

basic variable	x_1	x_2	x_3	S_1	S_2	b_i
C_i	-6	-4	-18	0	0	
x_3	1/3	0	1	-1/3	0	1
x_2	-2/3	1	0	2/3	-1	3

Because the Values party right are all positive values, it indicates that we've come to the optimal solution as follows:

$x_2 = 3 \quad x_3 = 1 \quad x_1 = S_1 = S_2 = 0 \quad Z = -36$

Example 2: Dual Simplex Method

EXAMPLE : Use Dual Simplex method to solve the LP problem.

$$\begin{aligned} \text{Maximize:} \quad & Z = -3X_1 - X_2 \\ \text{Subject to:} \quad & X_1 + X_2 \geq 1 \\ & 2X_1 + 3X_2 \geq 2 \\ & X_1, X_2 \geq 0 \end{aligned}$$

Step-1: Given problem is already in the form of maximization. So go to the next step.

Step-2: Convert the Given constraints into (\leq) form by multiplying with '-1' both sides.

$$\begin{aligned} \text{Maximize:} \quad & Z = -3X_1 - X_2 \\ \text{Subject to:} \quad & -X_1 - X_2 \leq -1 \\ & -2X_1 - 3X_2 \leq -2 \\ & X_1, X_2 \geq 0 \end{aligned}$$

Step-3: After introducing slack variables (S_1, S_2) standard form of the given problem is:

$$\begin{aligned} \text{Maximize:} \quad & Z = -3X_1 - X_2 + 0S_1 + 0S_2 \\ \text{Subject to:} \quad & -X_1 - X_2 + S_1 = -1 \\ & -2X_1 - 3X_2 + S_2 = -2 \\ & X_1, X_2, S_1, S_2 \geq 0 \end{aligned}$$

		X_1	X_2	s_1	s_2	
		-3	-1	0	0	
s_1 0		-1	-1	1	0	-1
s_2 0		-2	-3*	0	1	-2
		3	1	0	0	
Ratio		3/2	1/3			

Entering Variable ↓

Departing Variable →

Most negative ←

Min Ratio ↑

Select the pivot from the work column

pivot must be negative

we get the ration for -2 and -3 the one with min ratio will be the pivot

since -3 give the min ratio then X_2 will be the entering variable

Example 2: Cont...

The pivot to be 1... we will make row operation divide the complete row R2 by -3

	X_1 -3	X_2 -1	s_1 0	s_2 0	
s_1 0	-1	-1	1	0	-1
X_2 -1	2/3	1	0	-1/3	2/3
	3	1	0	0	

Then to get 0 up the pivot R2+R2

	X_1 -3	X_2 -1	s_1 0	s_2 0	
s_1 0	-1/3	0	1	-1/3	-1/3
X_2 -1	2/3	1	0	-1/3	2/3
	7/3	0	0	1/3	

Entering Variable

Departing Variable

	X_1	X_2	s_1	s_2	
	-3	-1	0	0	
s_1 0	-1/3	0	1	-1/3*	-1/3
X_2 -1	2/3	1	0	-1/3	2/3
	7/3	0	0	1/3	
Ratio	7			1	

Most negative

Min Ratio

This is the final table where the solution is optimal feasible
 The soln is $X_1=0$, $X_2 =1$, $s_1 = 0$, $s_2 =1$,
 $Z=-1$

$R1 * -3$

	X_1	X_2	s_1	s_2	
	-3	-1	0	0	
s_2 0	1	0	-3	1	1
X_2 -1	2/3	1	0	-1/3	2/3

$R2 + 1/3 R1$

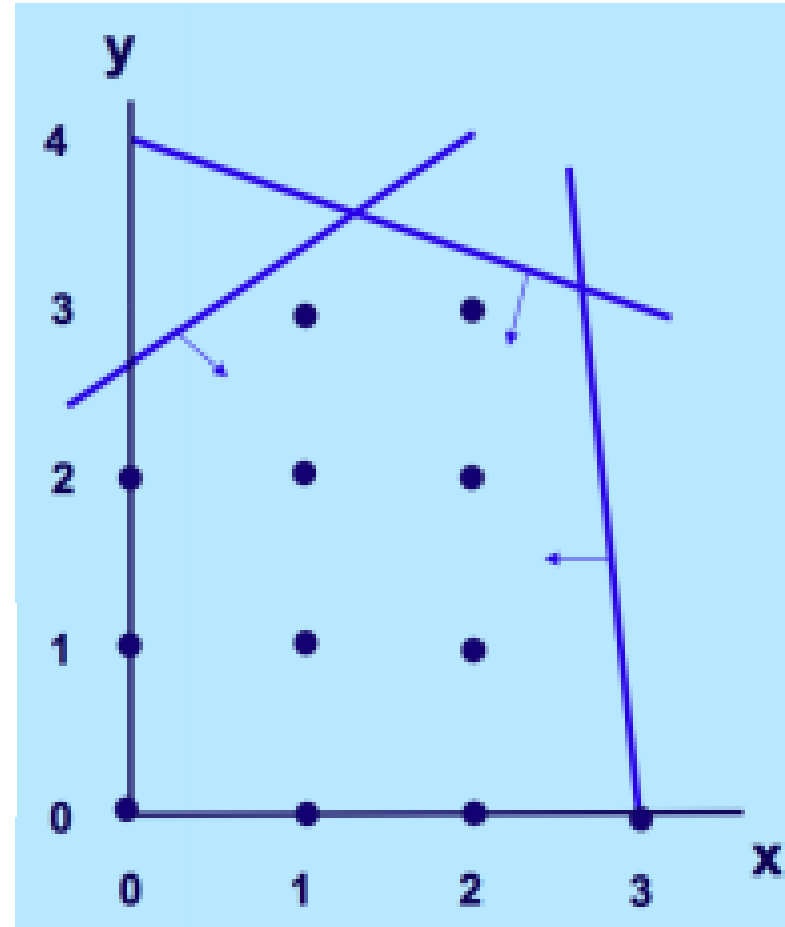
	X_1	X_2	s_1	s_2	
	-3	-1	0	0	
s_2 0	1	0	-3	1	1
X_2 -1	1	1	-1	0	1
	2	0	1	0	

Integer Programming

Some Facts

- Feasible region: a set of discrete points.
- Corner point solution not assured.
- No "efficient" way to solve an IP.

➤ Integer Linear Programming (ILP) models are much more difficult to solve than Linear Programming (LP) models.



Integer Programming

- In many practical problems, the decision variables actually make sense only if they have integer values.
- If some or all the decision variables in a linear programming formulation are required to have integer values, then it is an **Integer Programming (IP)** problem.
- The mathematical model for integer programming is the linear programming model with one additional restriction that some or all of the decision variables must have integer values.
- If only some of the decision variables are required to have integer values, then the model is called **Mixed Integer Programming (MIP)** problem.
- If all of the decision variables are required to have integer values, then the model is called **Pure Integer Programming** problem.

Maximize

$$Z = 2X_1 + X_2$$

Subject to

$$2X_1 + 5X_2 \leq 17$$

$$3X_1 + 2X_2 \leq 10$$

$$X_1, X_2 \geq 0 \text{ and integer}$$

Basic variable	X_1	X_2	S_1	S_2	RHS
	2	1	0	0	
S_1 0	0	11/3	1	-2/3	31/3
X_1 2	1	2/3	0	1/3	10/3
	0	1/3	0	2/3	

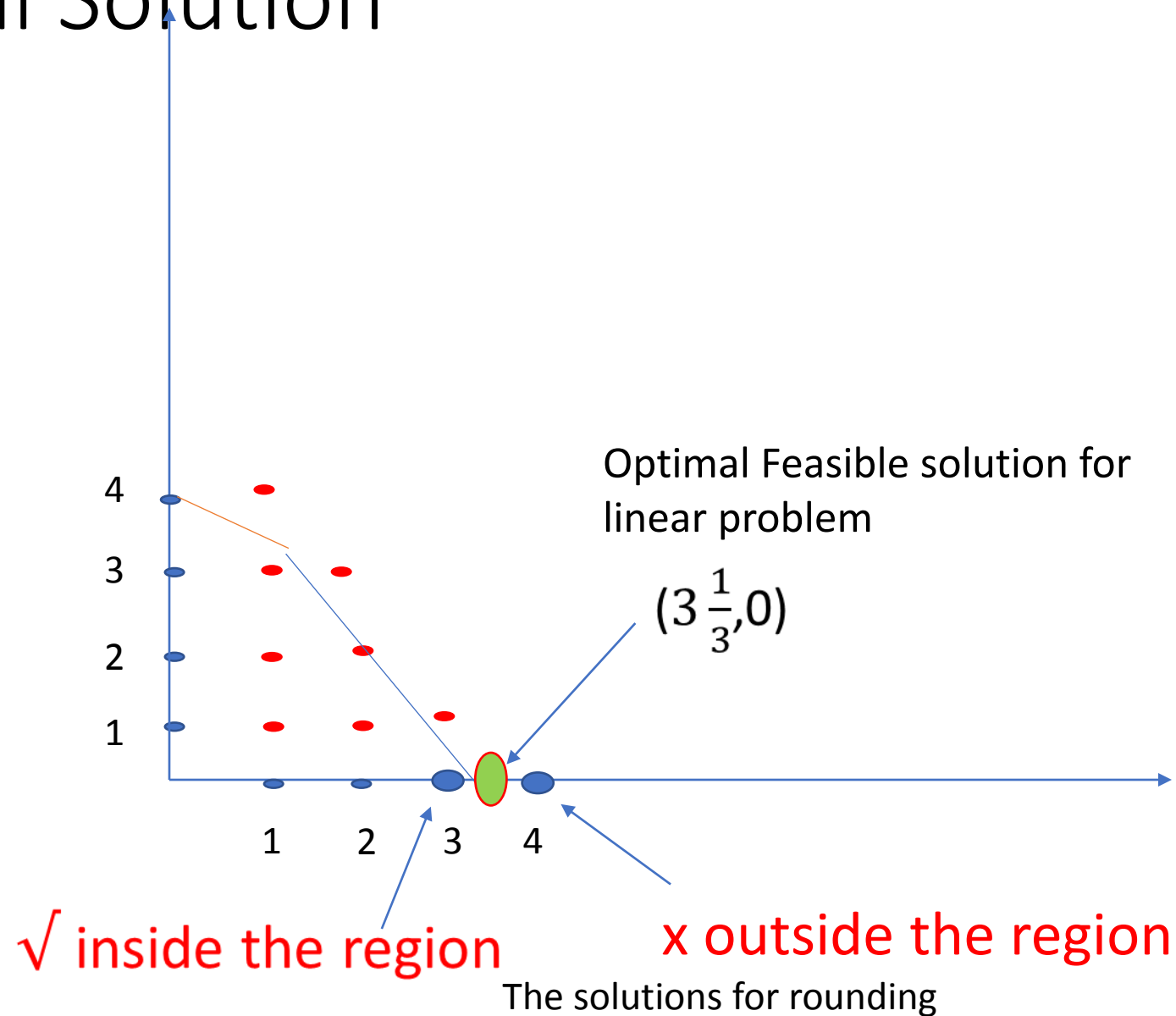
This the final optimal Tableau $X_1=3 \frac{1}{3}$ and $X_2=0$

- the final optimal real solution $X_1 = 3\frac{1}{3}$ and $X_2=0$

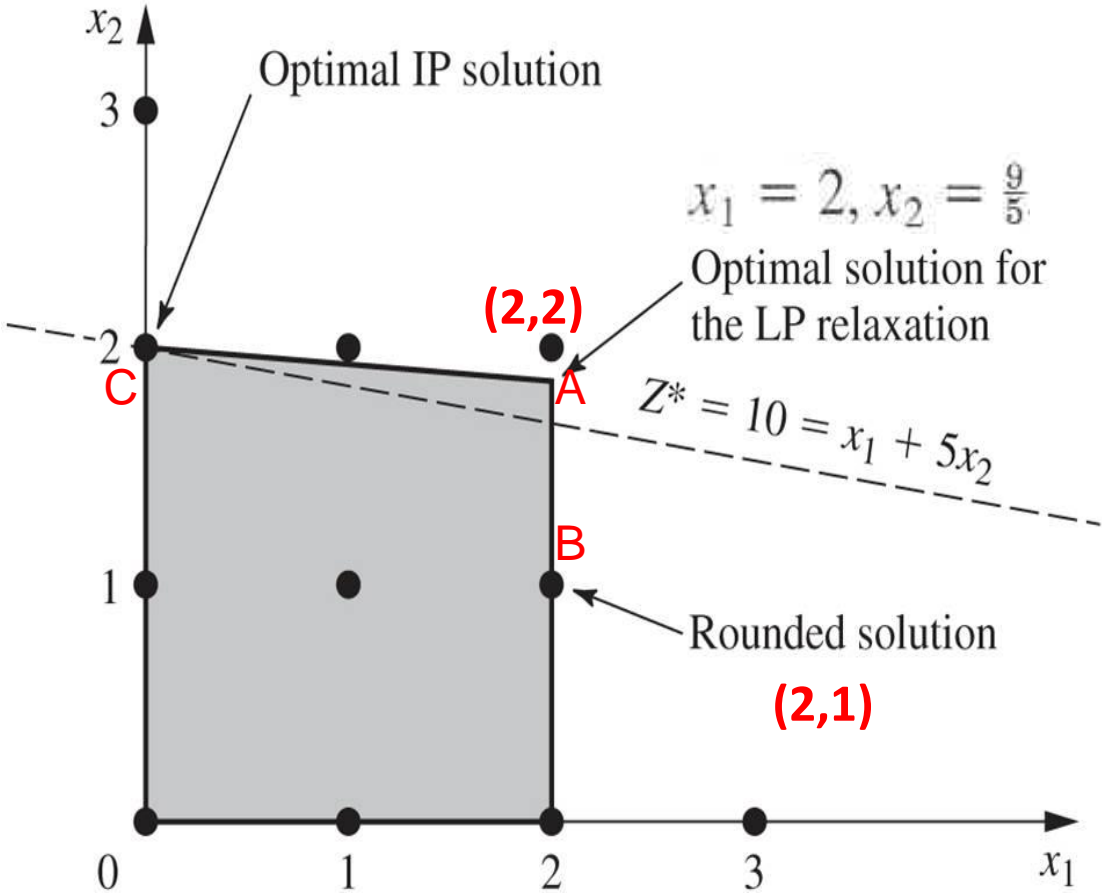
Which is infeasible for integer constraint

- By rounding X_1 up and down we obtain $(3,0)$ and $(4,0)$
- If we test these points if they satisfy the constraints or not we find that
- The first point $(3,0)$ satisfies the first and second constraint.
- while $(4,0)$ doesn't satisfy the second constraint
- So the solution for rounding Method is $(3,0)$

Graphical Solution



Maximize $Z = x_1 + 5x_2$,
 subject to
 $x_1 + 10x_2 \leq 20$
 $x_1 \leq 2$
 $x_1 \geq 0, \quad x_2 \geq 0$
 x_1, x_2 are integers.



Example 1: A Minimization Problem

- LP Formulation

$$\text{Min } z = 5x_1 + 2x_2$$

$$\text{s.t. } 2x_1 + 5x_2 \geq 10$$

$$4x_1 - x_2 \geq 12$$

$$x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

Example : Graphical Solution

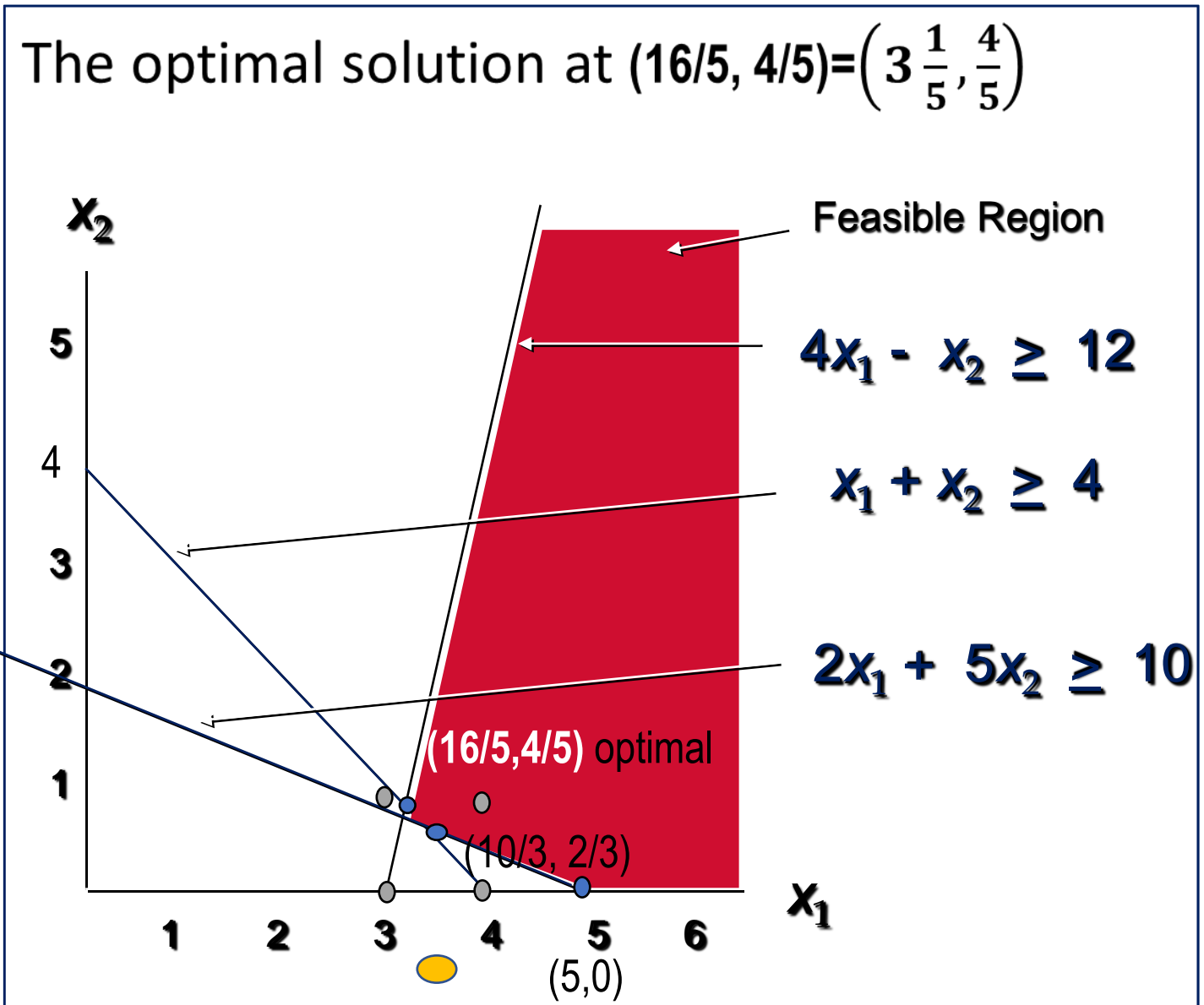
x_1 and x_2 are non-integer

By rounding x_1 and x_2 up and down we obtain the following solutions

$(3,0), (3,1), (4,1), (4,0)$

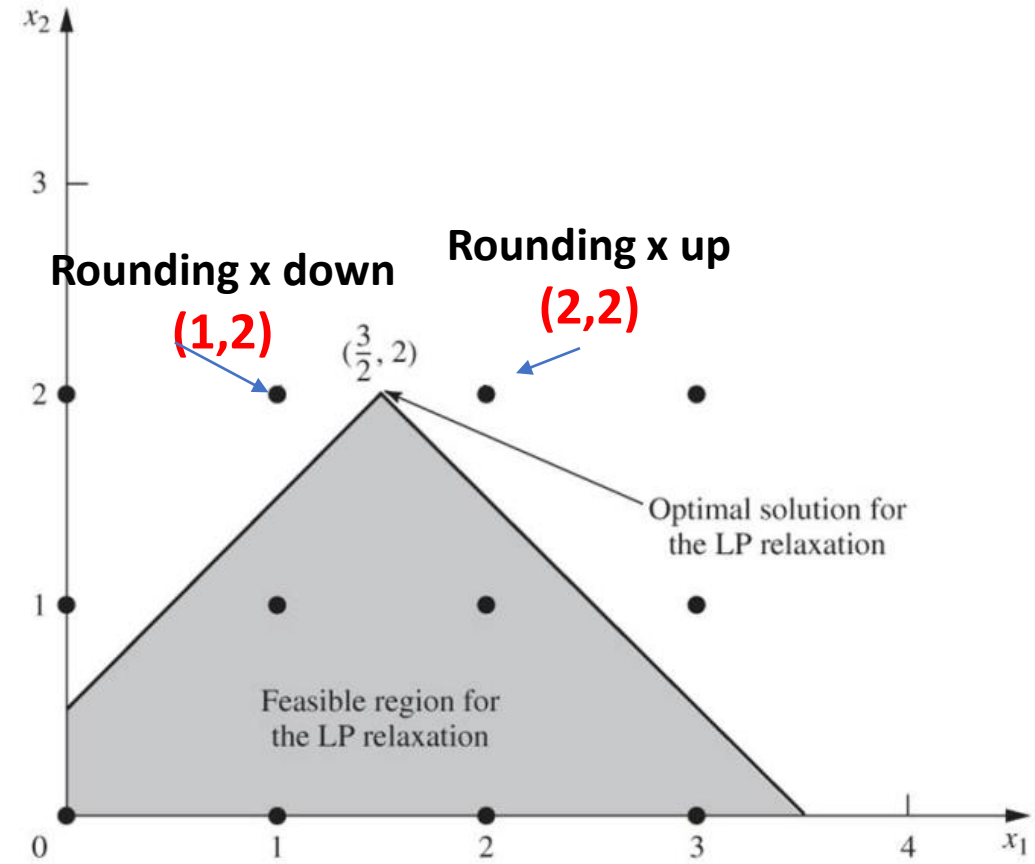
The only point that satisfy all the constraints is

$(4,1)$



2. Some Perspectives on Solving Integer Programming Problems

$$\begin{array}{ll} \text{Maximize} & Z = x_2, \\ \text{subject to} & \\ & -x_1 + x_2 \leq \frac{1}{2} \\ & x_1 + x_2 \leq 3\frac{1}{2} \\ & x_1 \geq 0, \quad x_2 \geq 0 \\ & x_1, x_2 \text{ are integers.} \end{array}$$



Rounding Fails to find the integer Solution

Because the optimal real is (1.5, 2) so after rounding up and down we obtain (1,2), (2,2) which are both outside the Region of Feasible solutions

Complexities of ILPS

- If an integer model is solved as a simple linear model, at the optimal solution non-integer values may be attained.
- Rounding to integer values may result in:
 - Infeasible solutions
 - Feasible but not optimal solutions
 - Optimal solutions.

Question 1: Pure IP problems have a finite number of feasible solutions. Is it possible to solve pure IP problems by exhaustive enumeration ?

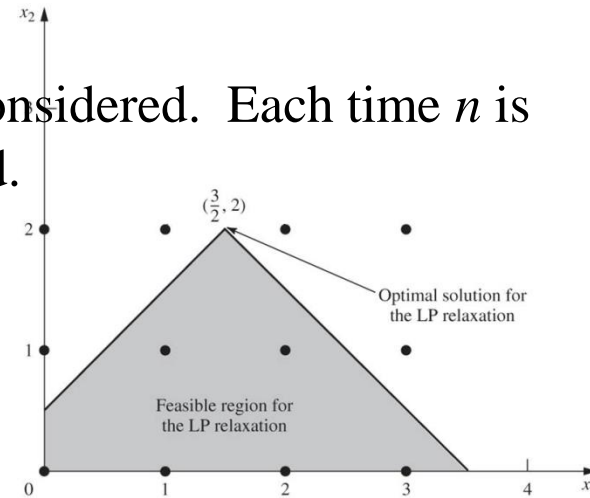
Consider the simple case of BIP problems.

With n variables, there are 2^n solutions to be considered. Each time n is increased by 1, the number of solutions is doubled.

$n=10$, 1024 solutions

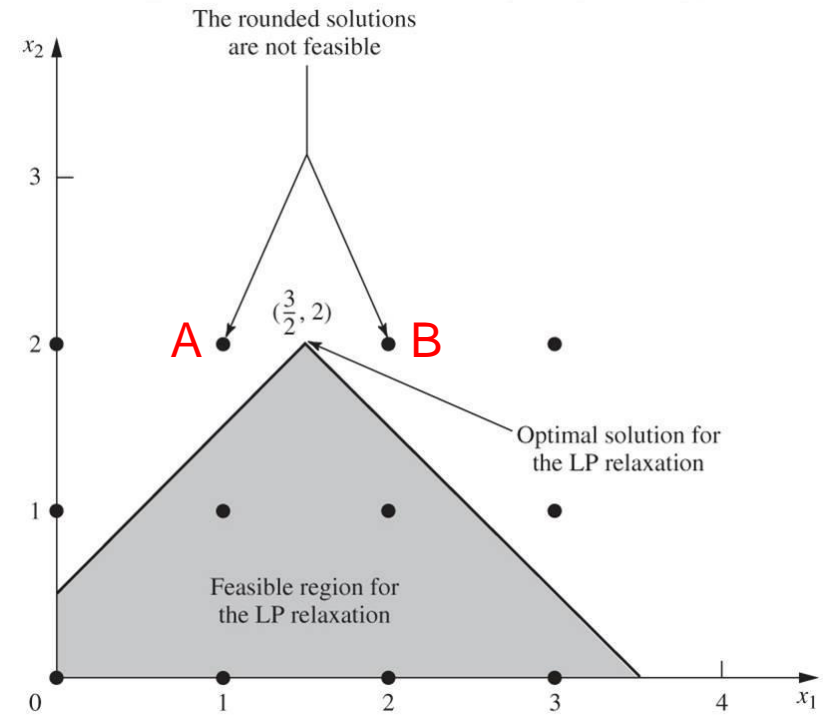
$n=20$, more than 1 million solutions

$n=30$; more than 1 billion solutions.



Question 2: Is IP easier to solve than LP because IP tends to have much fewer feasible solutions than LP ?

Question 3: Can we use the approximate procedure of simply applying the simplex method to get an LP optimal solution and then rounding the noninteger values of the solution to integers ?

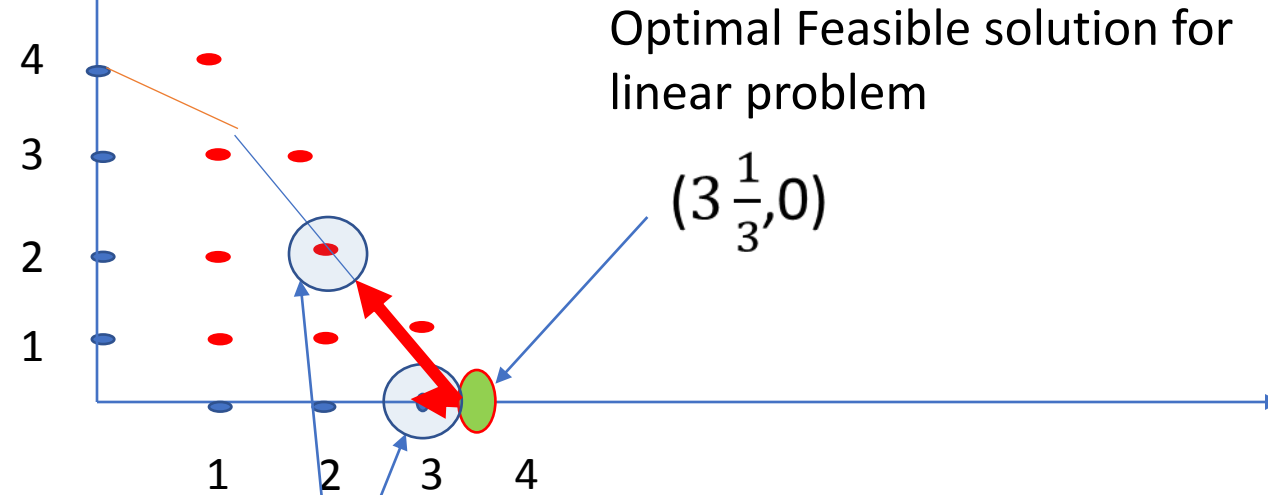


ε -close method

ε -close method

Trace all the constraints out from the optimal solution and the first integer point detected will be the optimal integer

In this graph the detected points will be $(2,2)$ with $Z = 2X_1 + X_2 = 6$
And $(3,0)$ with $Z = 2X_1 + X_2 = 6$
while rounding get only one solution



Optimal Feasible solution for linear problem

$(3 \frac{1}{3}, 0)$

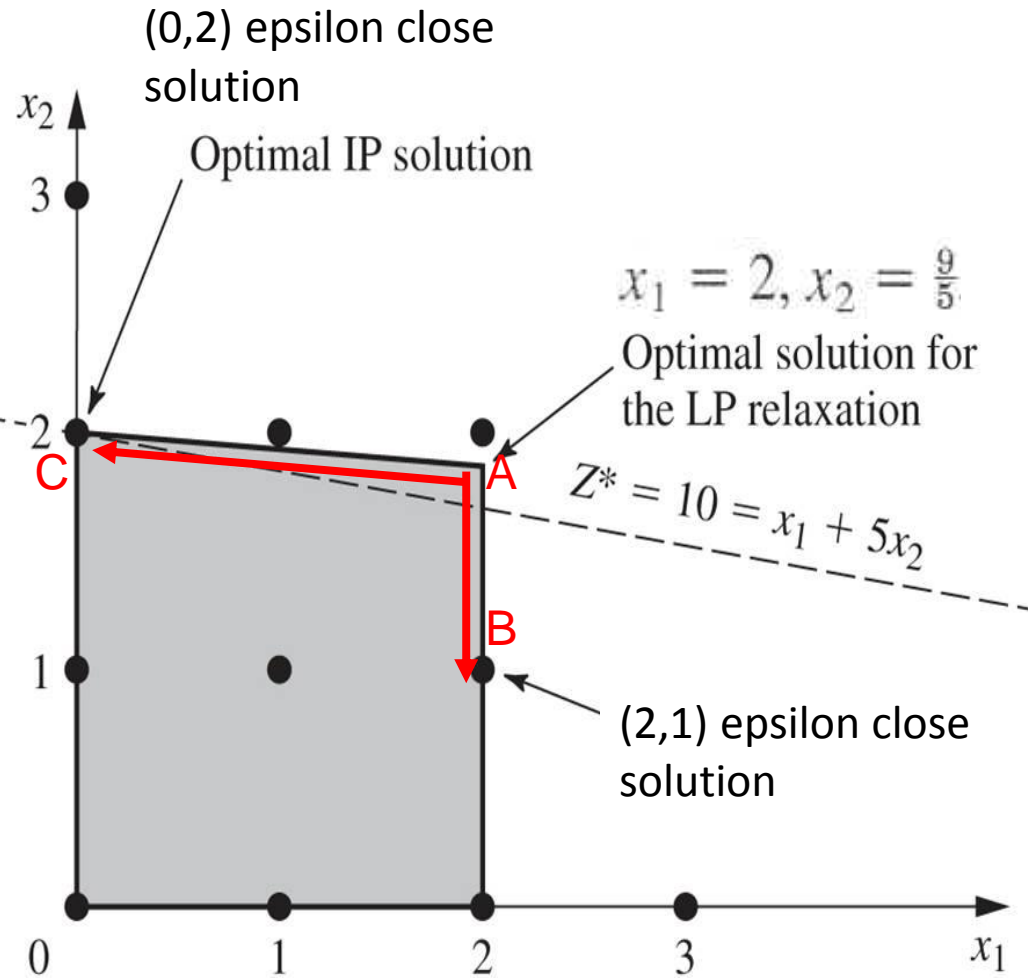
solution the region $(2,2)$ and $(3,0)$ are the integer solutions for epsoln close method

Maximize $Z = x_1 + 5x_2$,
 subject to
 $x_1 + 10x_2 \leq 20$
 $x_1 \leq 2$
 $x_1 \geq 0, \quad x_2 \geq 0$
 x_1, x_2 are integers.

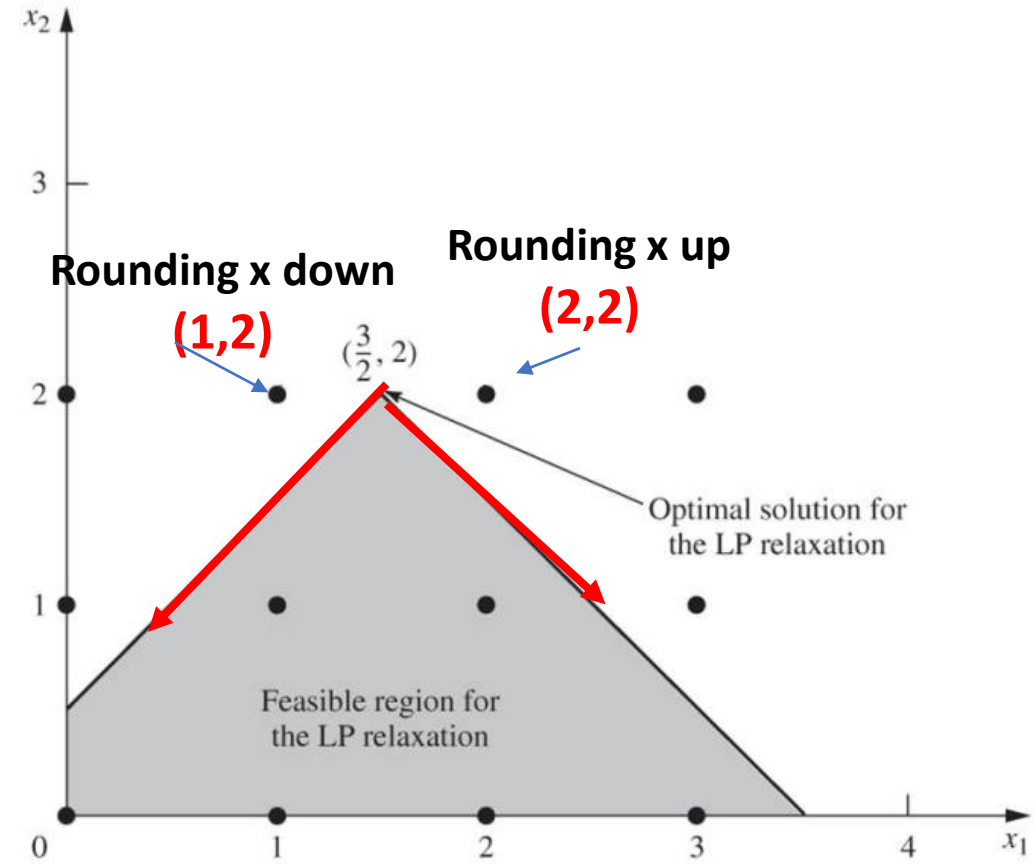
ϵ - close method

Trace all the constraints out from the optimal solution and the first integer point detected will be the optimal integer

In this graph the detected points will be (2,1) with $Z = x_1 + 5x_2 = 7$
 And **(0,2) with $Z = x_1 + 5x_2 = 10$**
 which is the optimal for IP



Maximize $Z = x_2$,
 subject to
 $-x_1 + x_2 \leq \frac{1}{2}$
 $x_1 + x_2 \leq 3\frac{1}{2}$
 $x_1 \geq 0, \quad x_2 \geq 0$
 x_1, x_2 are integers.



ϵ –close method **Fails to find the integer Solution**

Because along the two constraints that out from the optimal solution there is no integer point ...