Operations Research

Lecture 1: Introduction

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- **Textbook**: Problems in Operations Research, Gupta, 10^{th} edition
- Operations Research, Shaum
- **Grade**: 20% Quizes and Assignment + 20% mid-term exam + 60% final exam
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- **Course materials**:

  Textbook + PPT/Board + Lecture notes

Introduction

- **Operations Research** is an Art and Science
- **Operations Research** is the scientific approach to execute decision making, which consists of:
  - The art of ***mathematical modeling*** of complex situations
  - The science of the development of ***solution techniques*** used to solve these models

What is Operations Research?

- **OR** has its early roots in World War II and then in business and industry with the aid of computer
- Application areas of **OR** include military, industry, business, public sector, healthcare…

- Operational Research as termed in Europe, is an Branch of applied mathematics that uses techniques and statistics to arrive at Optimal solutions to solve complex problems

What is an optimal solution?

- An **optimal solution** is a feasible **solution** where the objective function reaches its maximum (or minimum) value – for example, the most profit or the least cost.
- A globally **optimal solution** is one where there are no other feasible **solutions** with better objective function values.

INTRODUCTION TO OPERATIONAL RESEARCH

Operational Research: is a systematic and analytical approach to decision making and problem solving.

It is typically concerned with determining the maximum profit, sale, output, crops yield and efficiency

And minimum losses, risks, cost, and time of some objective function.

USE OF COMPUTERS

The models of OR need lot of computation and therefore, the use of computers becomes necessary.

With the use of computers it is possible to handle complex problems requiring large amount of calculations.

The objective of the operations research models is to attempt and to locate Best or Optimal Solution.

NATIONAL PLANNING AND BUDGETING

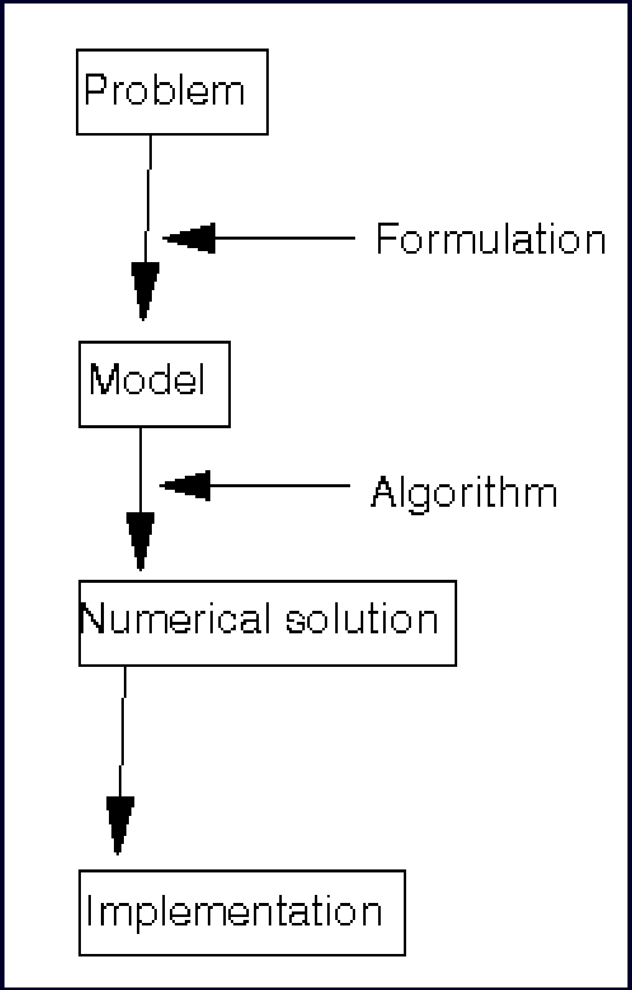
OR is used for the Preparation of-

- **Five Year Plans**
- **Annual Budgets**
- **Forecasting of Income and Expenditure**
- **Scheduling of Major Projects of National Importance**
- **Population**
- **Employment and Generation of Agriculture Yields, etc.**

1.3 An OR study: the general steps

An OR study usually contains the following steps:

1. Identify the problem;
2. Formulate the problem as a math model;
3. Develop effective algorithms
   or computer packages to solve the model;
5. Obtain numerical solution for the model;
6. Implementation



Course plan (preliminary)

Introduction

Chapter 1: Linear Programming: modeling
and graphical solution;

Chapter 2: ***Powerful Method***, LP: simplex method;
How to be efficient? LP: duality Theory
(Dual Simplex Method- Dual Problem)

Chapter 3: What if?
sensitive analysis;

Chapter 4: Integer Programming
1- epsilon close method   2- Rounding
3- Assigning non-zero value to the non-Basic variable

Chapter 5: Integer program Applications
(Transportation – Production- Transshipment- assignment)

Example: Iron Works.

Iron Factory, manufactures two products made from steel and just received this month's allocation of *2000* pounds of steel. It takes *2* pounds of steel to make a unit of product 1 and it takes *3* pounds of steel to make a unit of product 2.

unit profits for products 1 and 2, are*100* and *200* respectively.

The manufacturer has a contract calling for at least *60* units of product 1 this month. The firm's facilities are such that at most *720* units of product 2 may be produced monthly.

Example: Iron Works, Inc.

- Decision Variable: Let x_1 and x_2 denote this month's production level of product 1 and product 2, respectively.
- Objective:
  - The total monthly profit =

(profit per unit of product 1)

x (monthly production of product 1)

+ (profit per unit of product 2)

x (monthly production of product 2)

$= \ 100x_1 + 200x_2$

We want to maximize total monthly profit:

Max : Z= $100x_1 + 200x_2$

Example: Iron Works, Inc.

- Constraints
  - The total amount of steel used during monthly production =

(steel required per unit of product 1)

x (monthly production of product 1)

\+ (steel required per unit of product 2)

x (monthly production of product 2)

$$= 2x_1 + 3x_2$$

This quantity must be less than or equal to the allocated *b* pounds of steel:

$$2x_1 + 3x_2 \leq 2000$$

Example: Iron Works, Inc.

- Mathematical Model (continued)
  - The monthly production level of product 1 must be greater than or equal to *60*:

$$x_1 \geq 60$$

  - The monthly production level of product 2 must be less than or equal to *720*:

$$x_2 \leq 720$$

  - However, the production level for product 2 cannot be negative:

$$x_1 \text{ and } x_2 \geq 0$$

Example: Iron Works.

- Answer:

Substituting, the model is:

$$\text{Max: } Z = 100x_1 + 200x_2$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 + 3x_2 &\leq 2000 \\ x_1 &\geq 60 \\ x_2 &\leq 720 \\ x_1 \text{ and } x_2 &\geq 0 \end{aligned}$$

Example: Iron Works, Inc.

- Question:

  The optimal solution to the current model is $x_1 = 60$ and $x_2 = 626\ 2/3$. If the product were engines, explain why this is not a true optimal solution for the "real-life" problem.

- Answer:

  One cannot produce and sell 2/3 of an engine. Thus the problem is further restricted by the fact that both x_1 and x_2 must be integers. They could remain fractions if it is assumed these fractions are work in progress to be completed the next month.

Example: Iron Works

Inputs

$100 profit per unit Prod. 1
$200 profit per unit Prod. 2
2 lbs. steel per unit Prod. 1
3 lbs. Steel per unit Prod. 2
2000 lbs. steel allocated
60 units minimum Prod. 1
720 units maximum Prod. 2
0 units minimum Prod. 2

Controllable Inputs

60 units Prod. 1
626.67 units Prod. 2

Mathematical Model

$$
\begin{array}{llll}
\text{Max} & 100(60) + 200(626.67) & & \\
\text{s.t.} & 2(60) + 3(626.67) & \leq & 2000 \\
 & 60 & \geq & 60 \\
 & 626.67 & \leq & 720 \\
 & 626.67 & \geq & 0
\end{array}
$$

Output

Profit = $131,333.33
Steel Used = 2000

Graphical solution

A Graphical Solution Procedure (LPs with 2 decision variables can be solved/viewed this way.)

1. Plot each constraint as an equation and then decide which side of the line is feasible (if it's an inequality).
2. Find the feasible region.
3. find the coordinates of the corner (extreme) points of the feasible region.
4. Substitute the corner point coordinates in the objective function
5. Choose the optimal solution

Example 2: A Minimization Problem

- LP Formulation

$$\text{Min} \quad z = 5x_1 + 2x_2$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 + 5x_2 &\geq 10 \\ 4x_1 - x_2 &\geq 12 \\ x_1 + x_2 &\geq 4 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Example 1: Graphical Solution

- Graph the Constraints

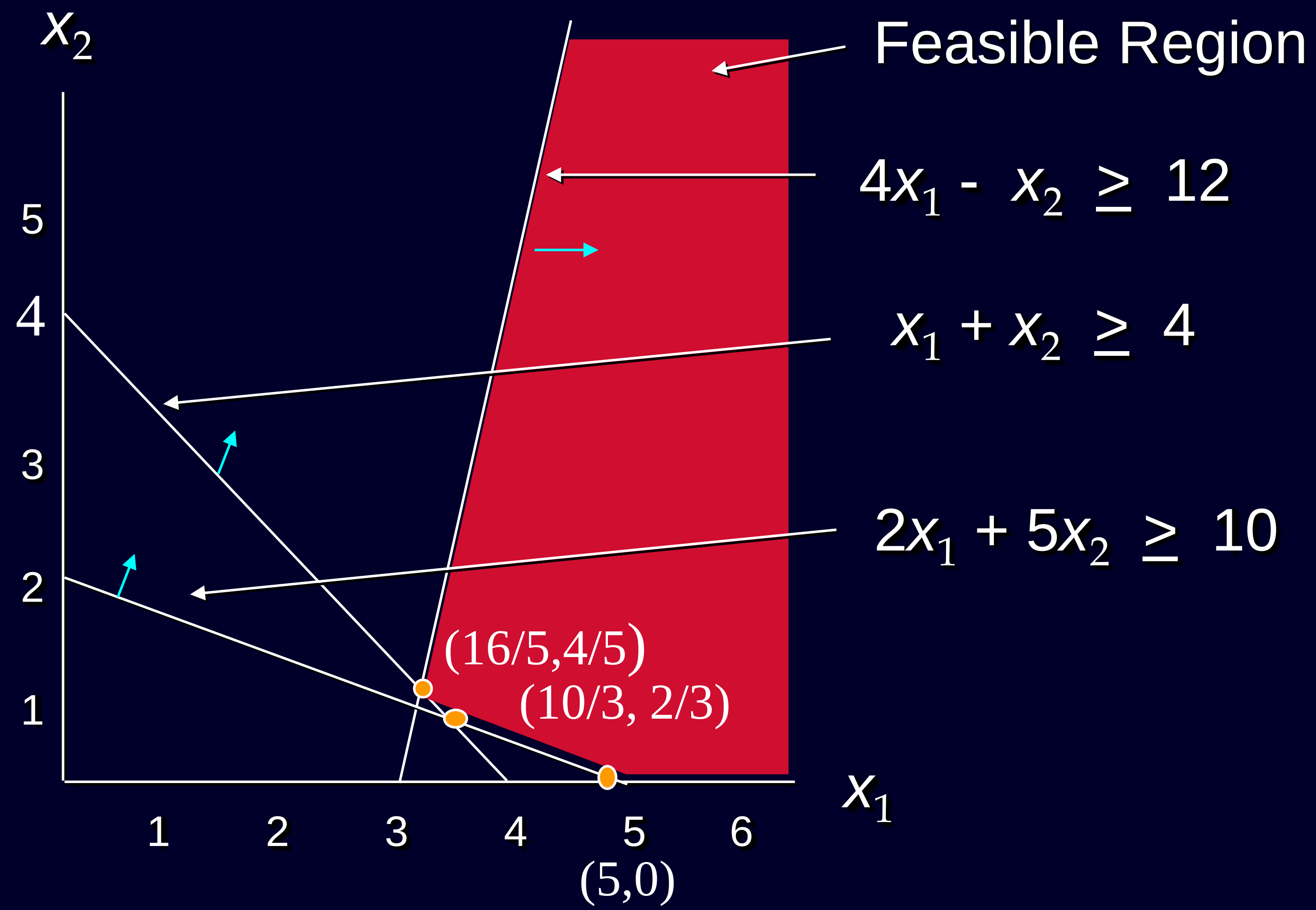
Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$. Connect (5,0) and (0,2). The ">" side is above this line.

Constraint 2: When $x_2 = 0$, then $x_1 = 3$. But setting x_1 to 0 will yield $x_2 = -12$, which is not on the graph. Thus, to get a second point on this line, set x_1 to any number larger than 3 and solve for x_2: when $x_1 = 5$, then $x_2 = 8$. Connect (3,0) and (5,8). The ">" side is to the right.

Constraint 3: When $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,4). The ">" side is above this line.

Example 1: Graphical Solution

- Constraints Graphed



Example 1: Graphical Solution

- Solve for the Extreme Point at the Intersection of the second and third Constraints

$$4x_1 - x_2 = 12$$
$$x_1 + x_2 = 4$$

Adding these two equations gives:

$$5x_1 = 16 \quad \text{or} \quad x_1 = 16/5.$$

Substituting this into $x_1 + x_2 = 4$ gives: $x_2 = 4/5$

- Solve for the extreme point at the intersection of the first and third constraints

$$2x_1 + 5x_2 = 10$$
$$x_1 + x_2 = 4$$

Multiply the second equation by -2 and add to the first equation, gives

$$3x_2 = 2 \text{ or } x_2 = 2/3$$

Substituting this in the second equation gives x1 = 10/3

point	Z
(16/5, 4/5)	88/5
(10/3, 2/3)	18
(5, 0)	25

Example 2: A Maximization Problem

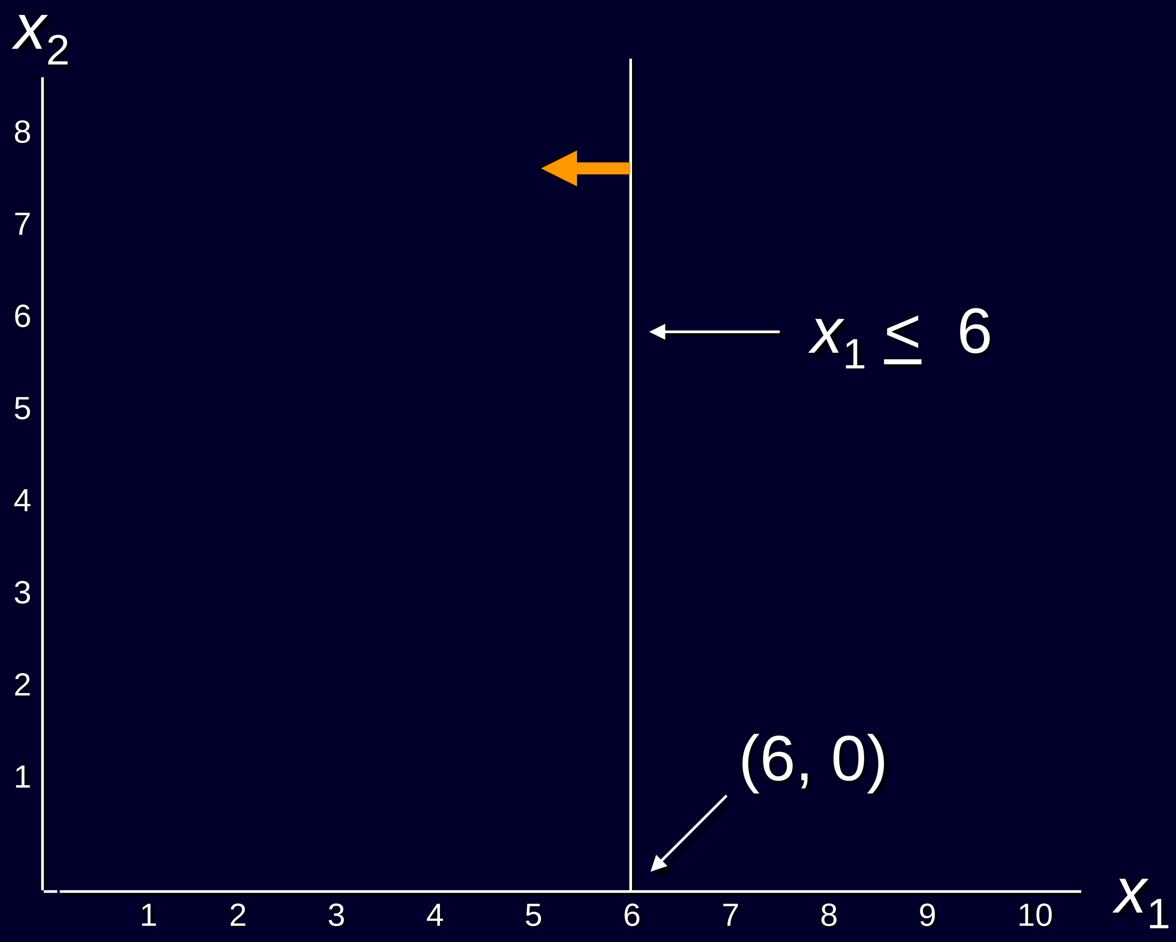
$$\text{Max} \quad z = 5x_1 + 7x_2$$

$$\begin{aligned} \text{s.t.} \quad x_1 \phantom{{}+3x_2} &\le 6 \\ 2x_1 + 3x_2 &\le 19 \\ x_1 + x_2 &\le 8 \end{aligned}$$

$$x_1, x_2 \ge 0$$

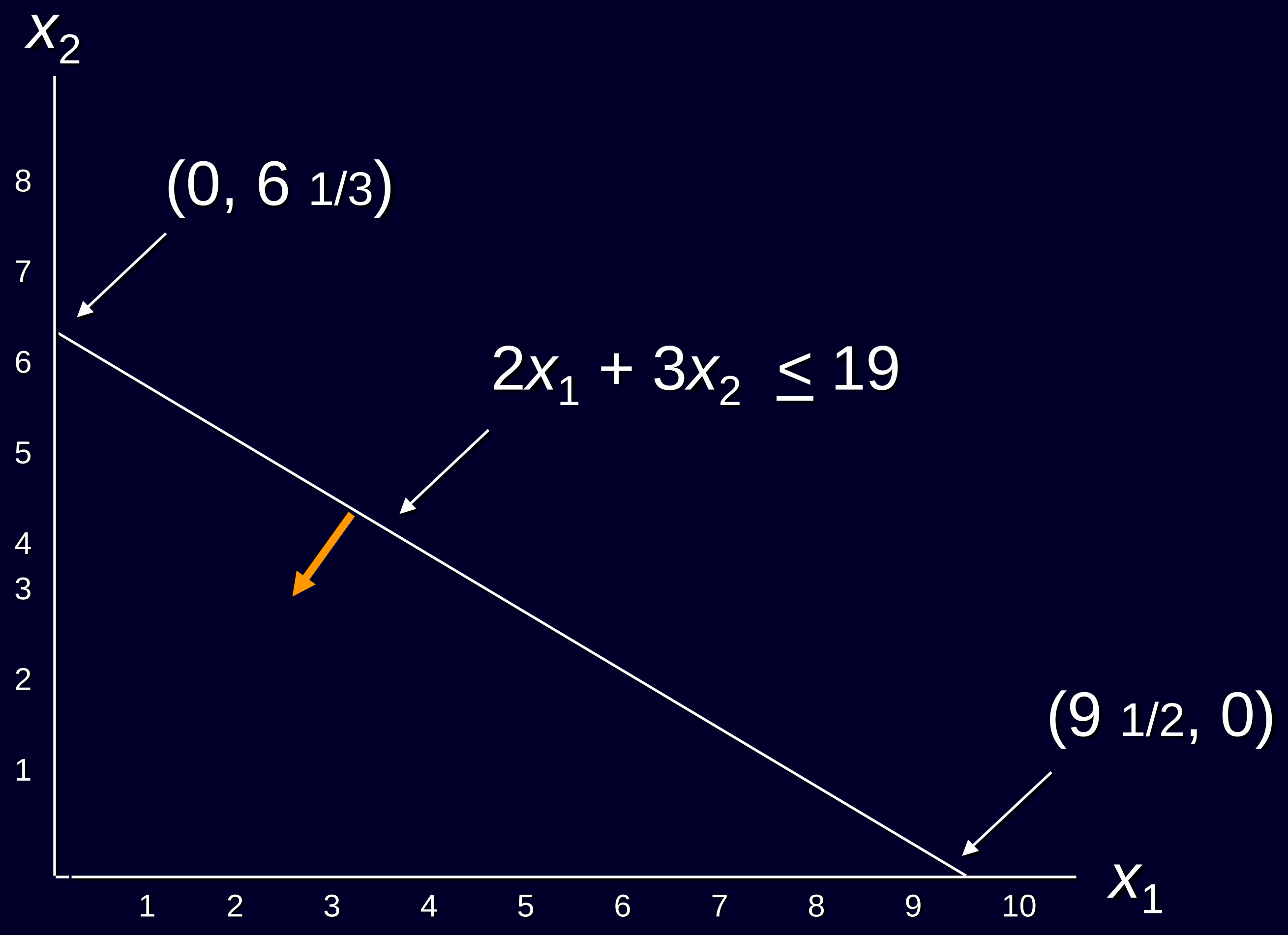
Example 2: A Maximization Problem

- Constraint #1 Graphed



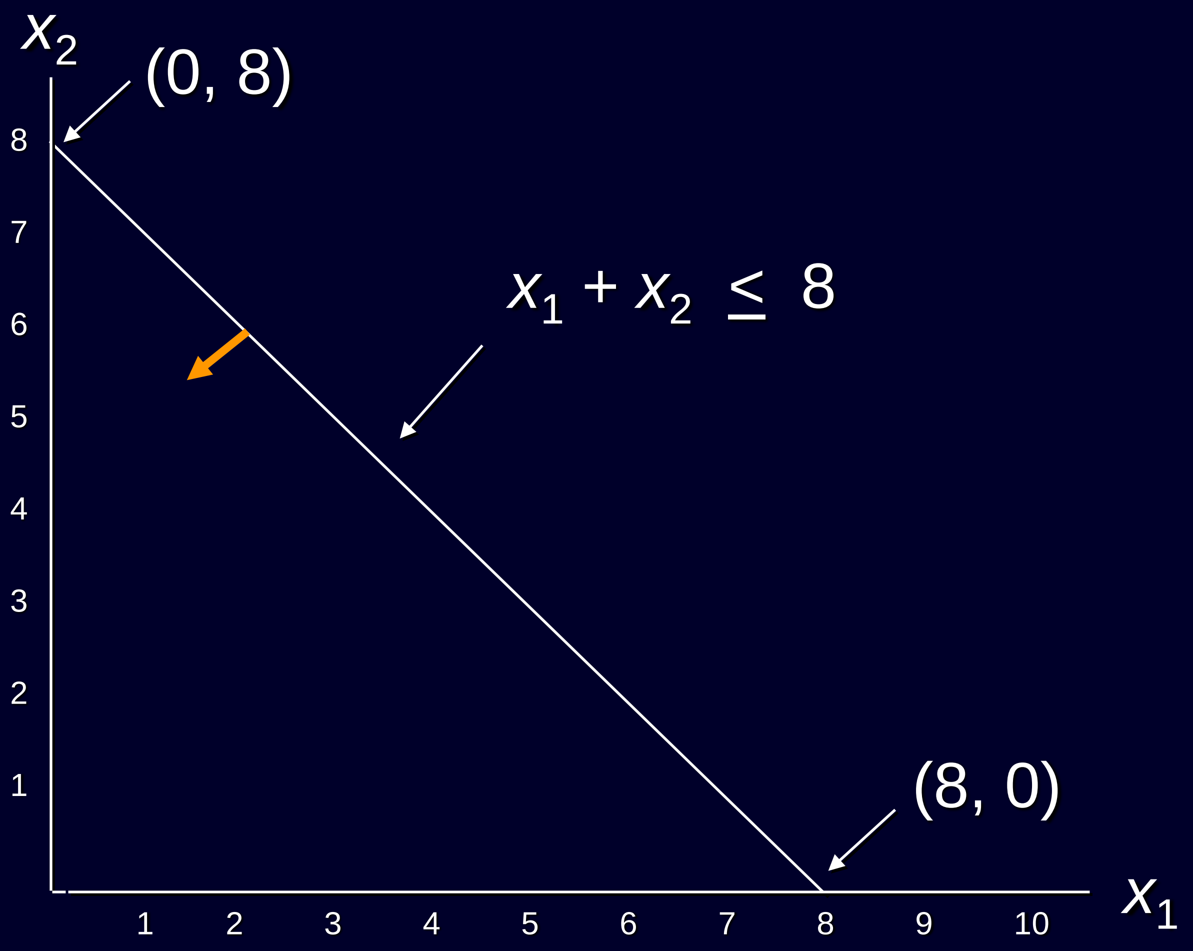
Example 2: A Maximization Problem

- Constraint #2 Graphed



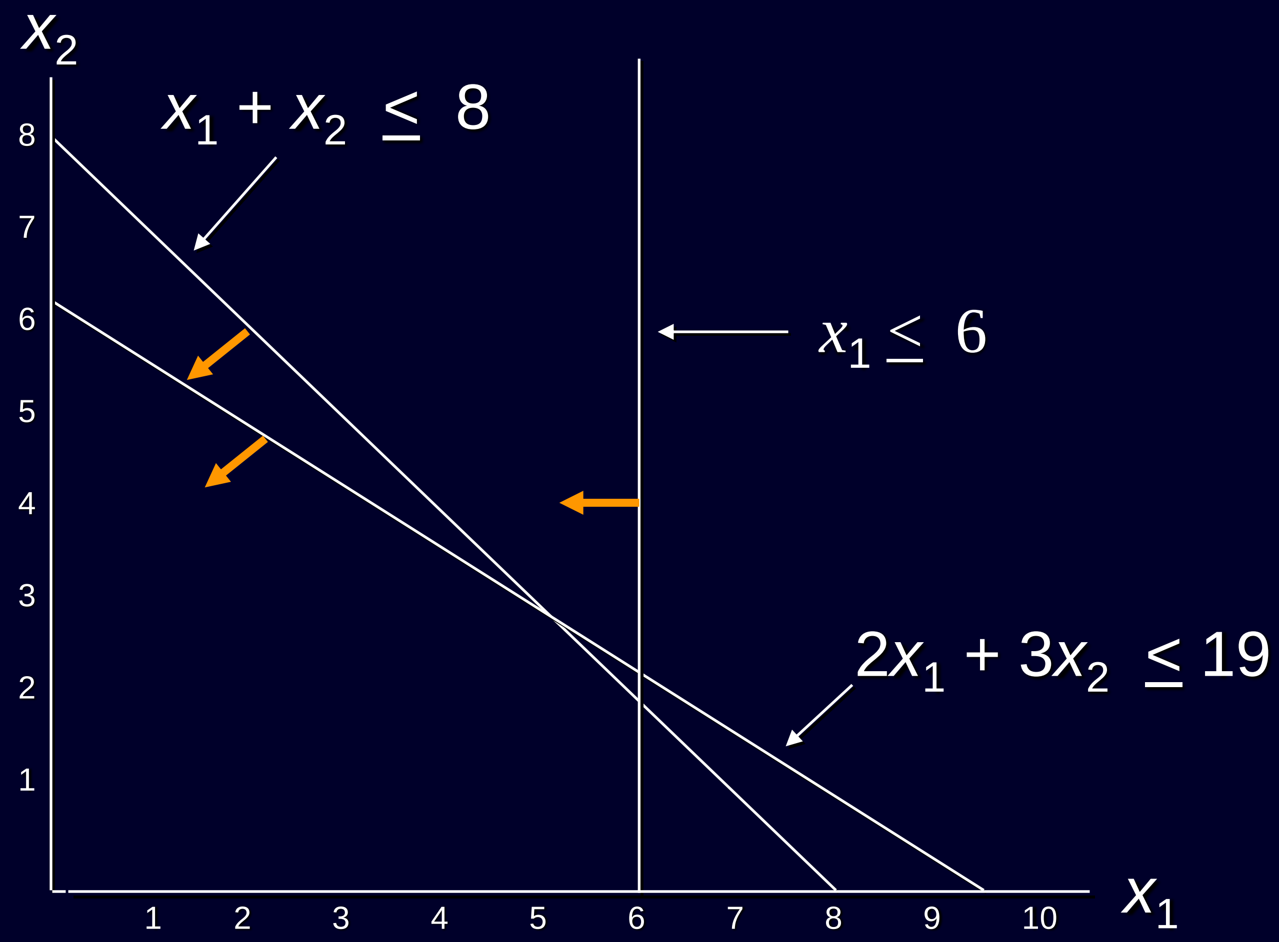
Example 2: A Maximization Problem

- Constraint #3 Graphed



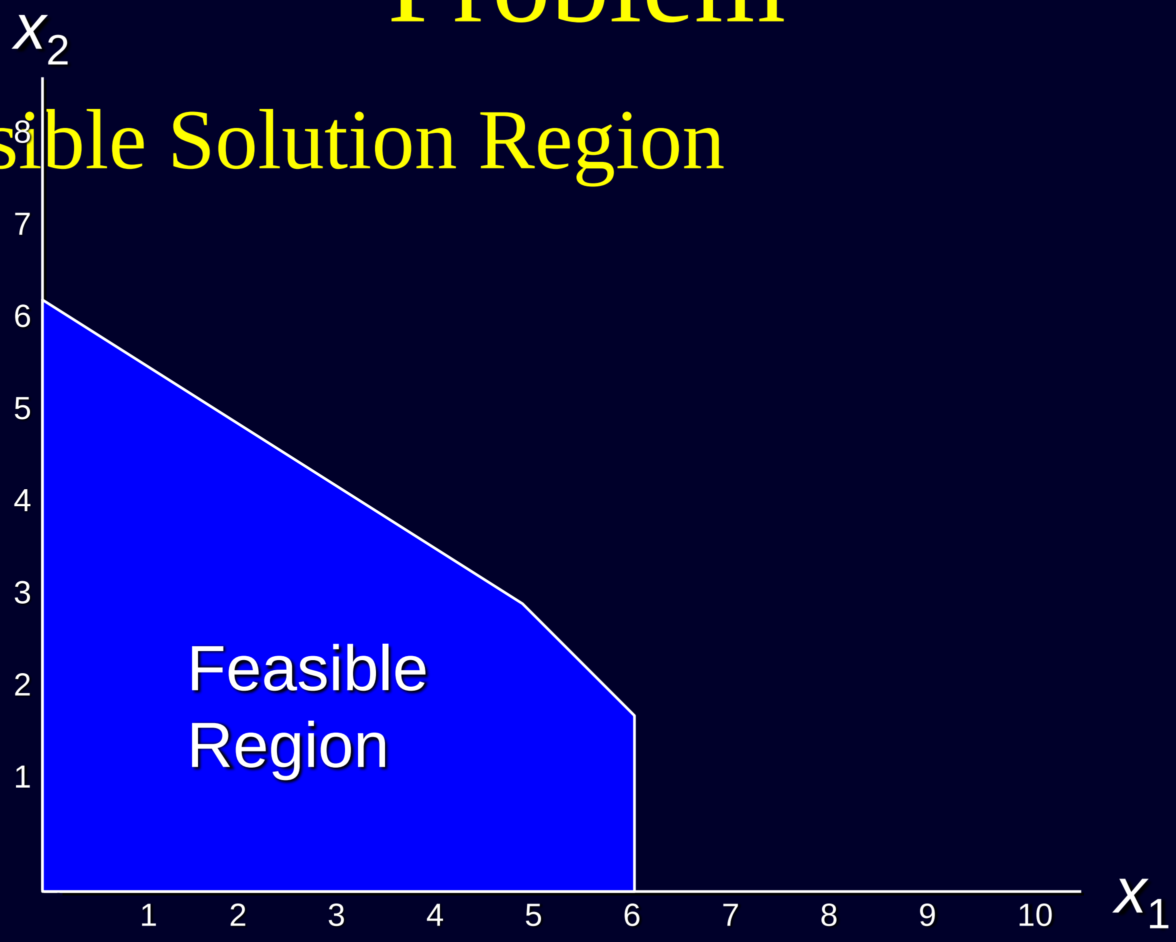
Example 2: A Maximization Problem

- Combined-Constraint Graph



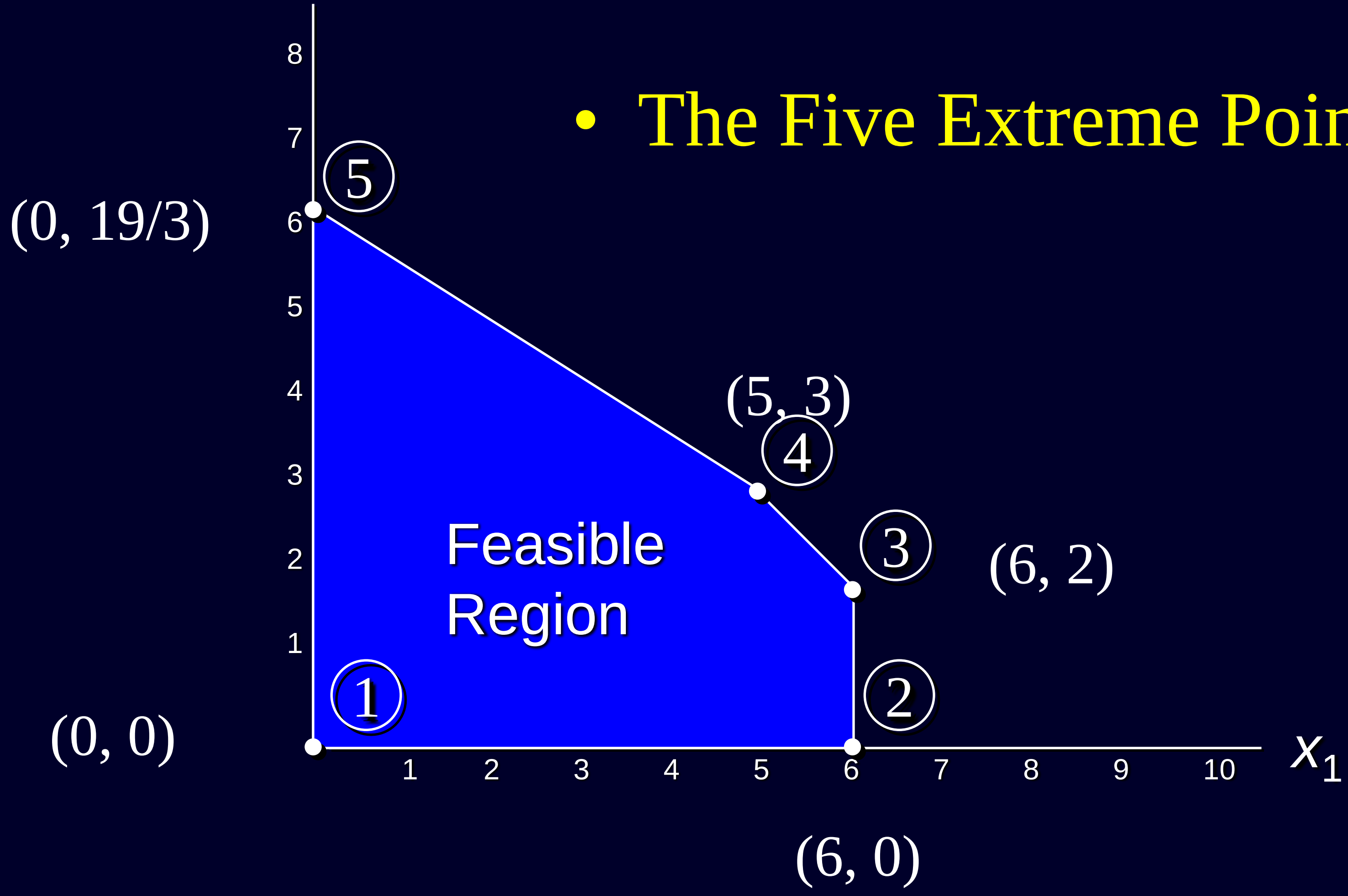
Example 2: A Maximization Problem

- Feasible Solution Region



Example 2: A Maximization Problem

- The Five Extreme Points

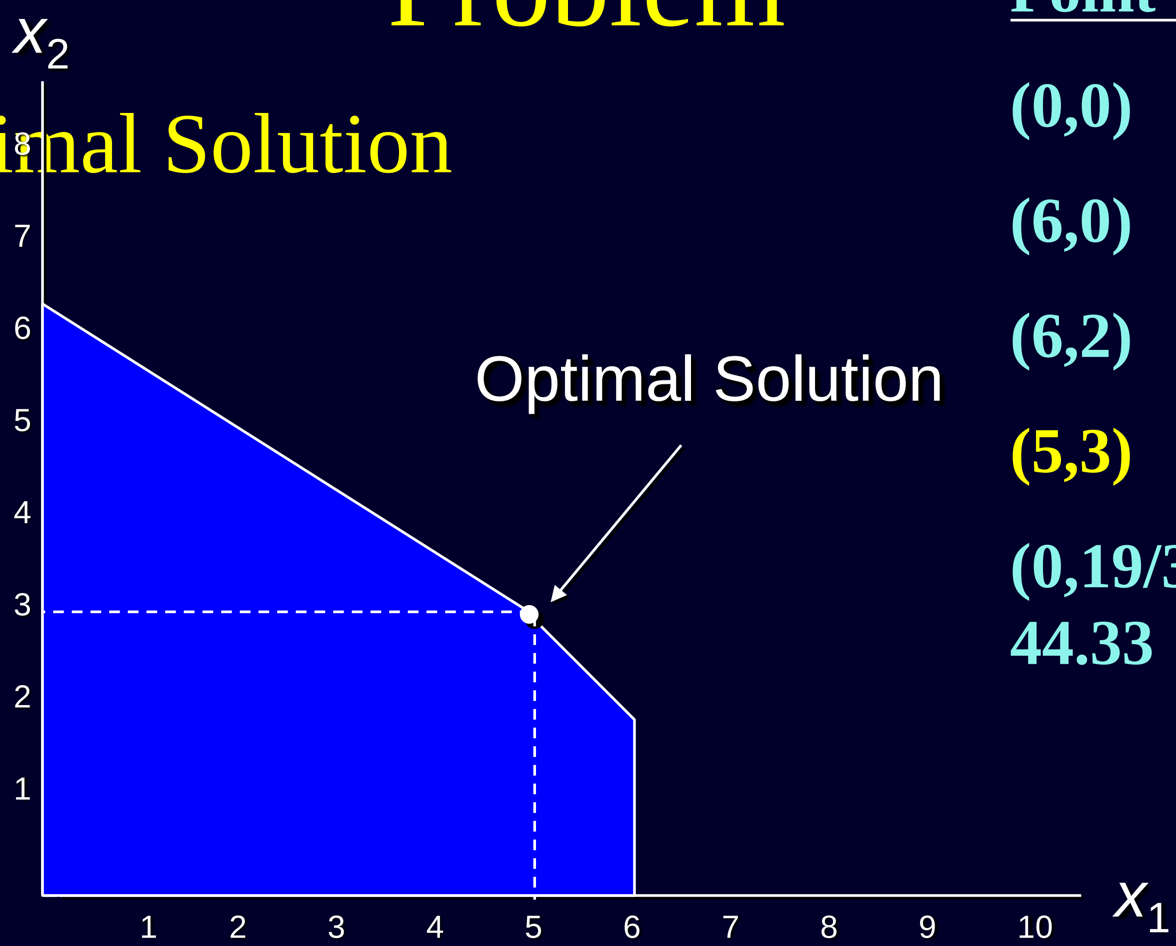


Example 2: A Maximization Problem

- Having identified the feasible region for the problem, we now search for the optimal solution, which will be the point in the feasible region with the largest (in case of maximization or the smallest (in case of minimization) of the objective function.
- To find this optimal solution, we need to evaluate the objective function at each one of the corner points of the feasible region.

Example 2: A Maximization Problem

- Optimal Solution



Point	Z
(0,0)	0
(6,0)	30
(6,2)	44
(5,3)	**46**
(0,19/3)	44.33

Extreme Points and the Optimal Solution

- The corners or vertices of the feasible region are referred to as the extreme points.
- An optimal solution to an LP problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- You have to consider only the extreme points of the feasible region.

Feasible Region

- The feasible region for a two-variable linear programming problem can be nonexistent, a single point, a line, a polygon, or an unbounded area.
- Any linear program falls in one of three categories:
  – is infeasible
  – has a unique optimal solution or alternate optimal solutions
  – has an objective function that can be increased without bound

Special Cases

- Alternative Optimal Solutions

  In the graphical method, if the objective function line is parallel to a boundary constraint in the direction of optimization, there are alternate optimal solutions, with all points on this line segment being optimal.

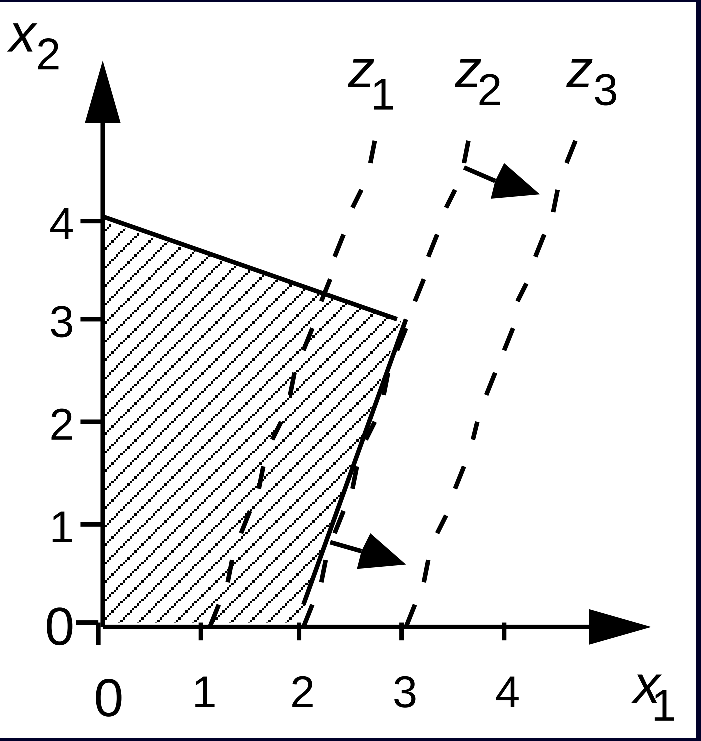
- Infeasibility (No Solution)

  A linear program which is over constrained so that no point satisfies all the constraints is said to be infeasible.

- Unbounded

  For a max (min) problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrarily large (small) Z

Example with Multiple Optimal Solutions



$$\text{Maximize} \quad z = 3x_1 - x_2$$

$$\text{subject to} \quad 15x_1 - 5x_2 \leq 30$$

$$10x_1 + 30x_2 \leq 120$$

$$x_1 \geq 0, \;\; x_2 \geq 0$$

Example: Infeasible Problem

- Solve graphically for the optimal solution:

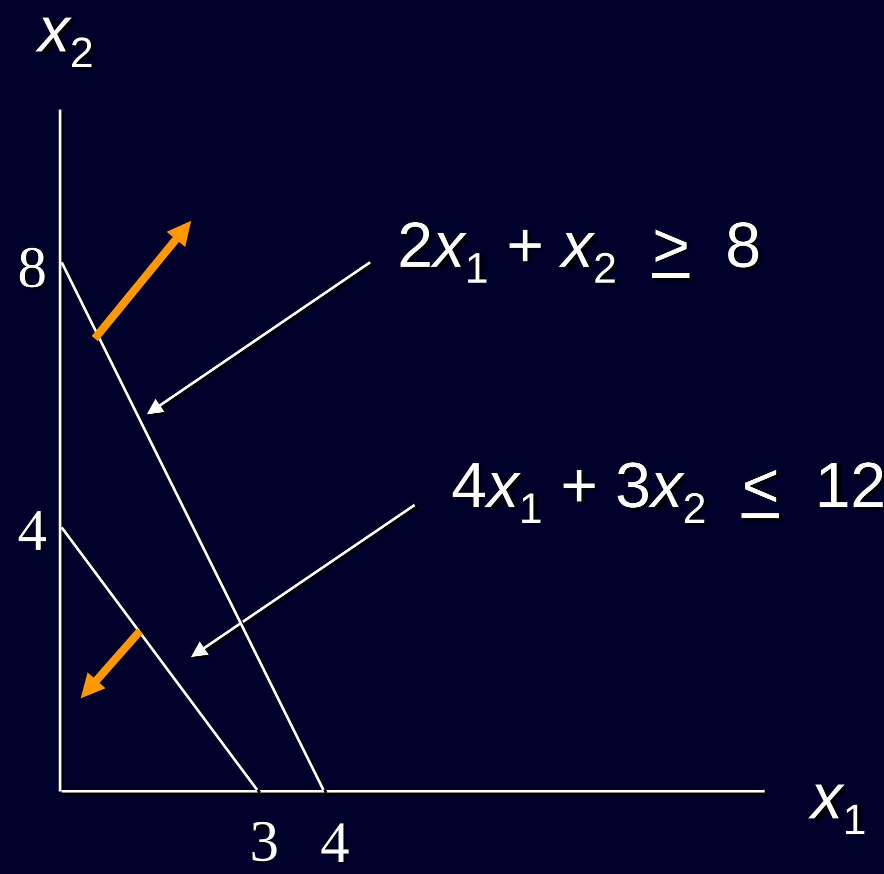
$$\text{Max} \quad z = 2x_1 + 6x_2$$

$$\begin{aligned} \text{s.t.} \qquad 4x_1 + 3x_2 &\leq 12 \\ 2x_1 + x_2 &\geq 8 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Example: Infeasible Problem

- There are no points that satisfy both constraints, hence this problem has no feasible region, and no optimal solution.



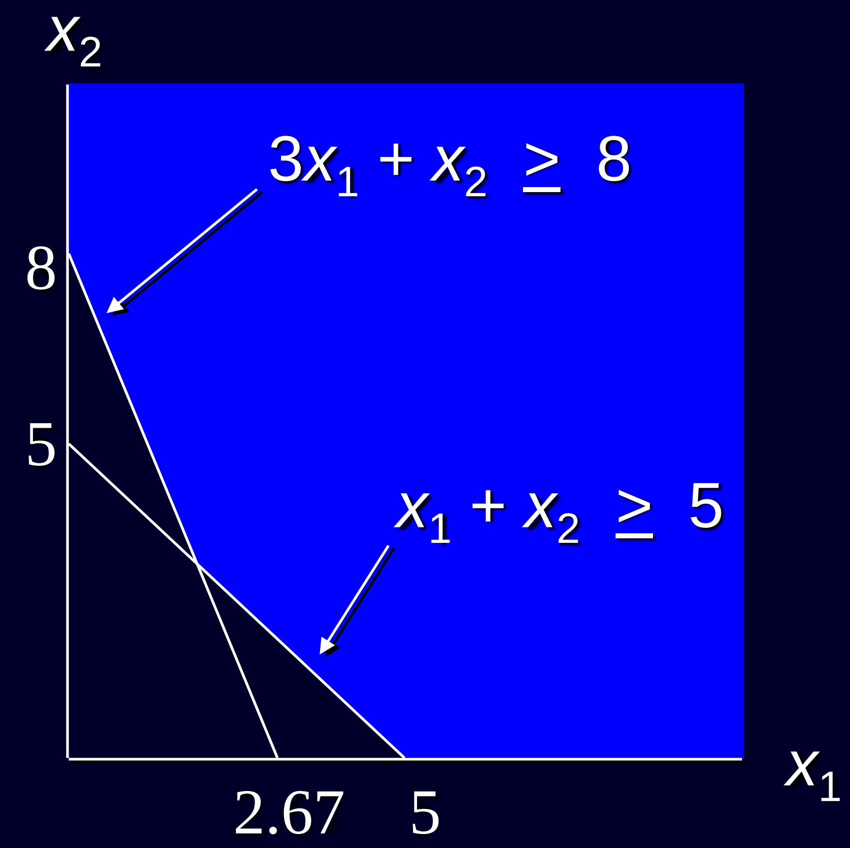
Example: Unbounded Problem

- Solve graphically for the optimal solution:

$$\begin{aligned} \text{Max} \quad & z = 3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 5 \\ & 3x_1 + x_2 \geq 8 \\ & x_1 , x_2 \geq 0 \end{aligned}$$

Example: Unbounded Problem

- The feasible region is unbounded and the objective function line can be moved parallel to itself without bound so that *z* can be increased infinitely.



Solve the following LP graphically

$$
\begin{array}{llll}
\max & 45\,x_1 + 60\,x_2 & & \\
\text{s.t.} & 10\,x_1 + 20\,x_2 \le 1800 & \text{A} & \\
 & 28\,x_1 + 12\,x_2 \le 1440 & \text{B} & \\
 & 6\,x_1 + 15\,x_2 \le 2040 & \text{C} & \\
 & 15\,x_1 + 10\,x_2 \le 2400 & \text{D} & \\
 & x_1 \le 40 & \text{E} & \\
 & x_2 \le 100 & \text{F} & \\
 & x_1 \ge 0,\ x_2 \ge 0 & &
\end{array}
$$

← Objective Function

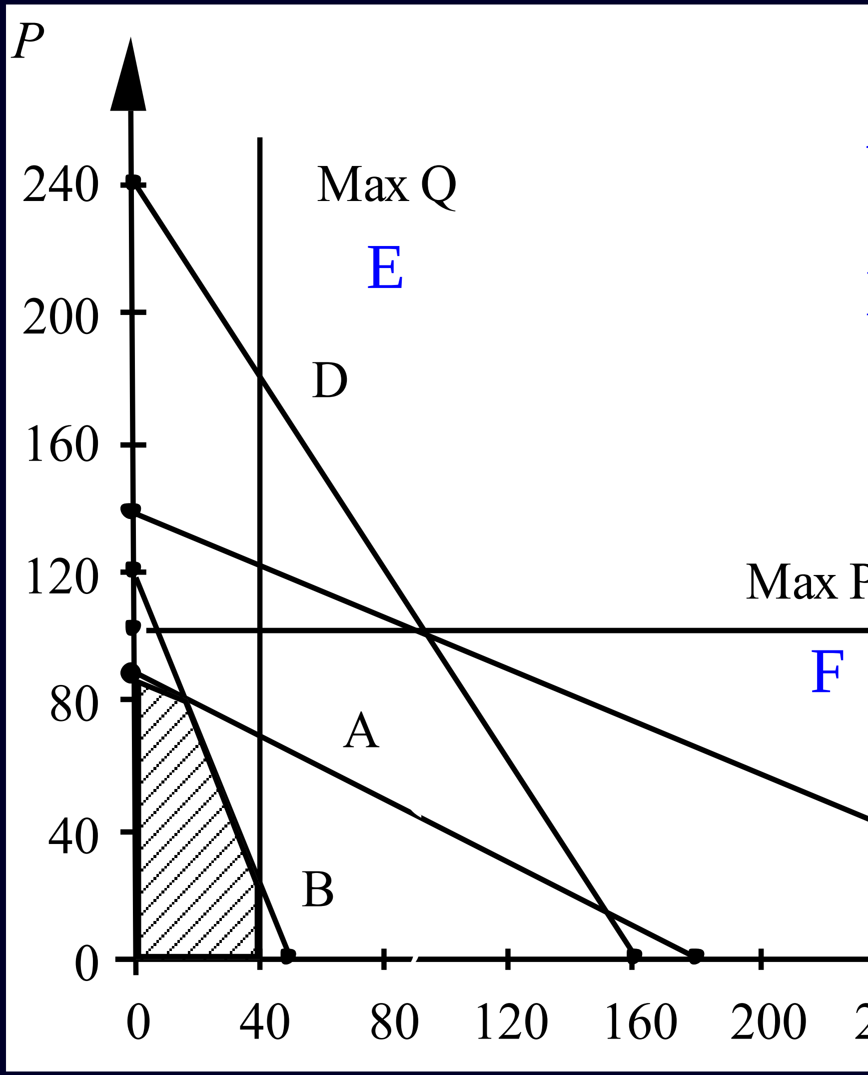
← Structural constraints

← nonnegativity

Where x_1 is the quantity produced from product Q, and x_2 is the quantity of product P

Are the LP assumptions valid for this problem?

The graphical solution



Assignment: Complete the problem to find the optimal solution

Possible Outcomes of an LP

1. Infeasible – feasible region is empty; e.g., if the constraints include $x_1 + x_2 \leq 6$ and $x_1 + x_2 \geq 7$
2. Unbounded - $\text{Max } 15x_1 + 7x_2$ (no finite optimal solution)
   $\text{s.t. } x_1 + x_2 \geq 1$
   $x_1, x_2 \geq 0$
3. Multiple optimal solutions - $\max\ 3x_1 + 3x_2$
   $\text{s.t. } x_1 + x_2 \leq 1$
   $x_1, x_2 \geq 0$
4. Unique Optimal Solution

Note: multiple optimal solutions occur in many practical (real-world) LPs.


Thank
You