Determinants

The determinant of a square matrix A is denoted by the symbol |A| or det A. We can form determinants of n x n matrices.

1. If $A = [a_{11}]$ is a 1×1 matrix, then its determinant $|A|$ is equal to the number a_{11} itself.

1. $|2| = 2$

2. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix, then the determinant is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

2. $\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = 2 \times (-5) - 3 \times 4 = -10 - 12 = -22$

3. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix, then its determinant is given by

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

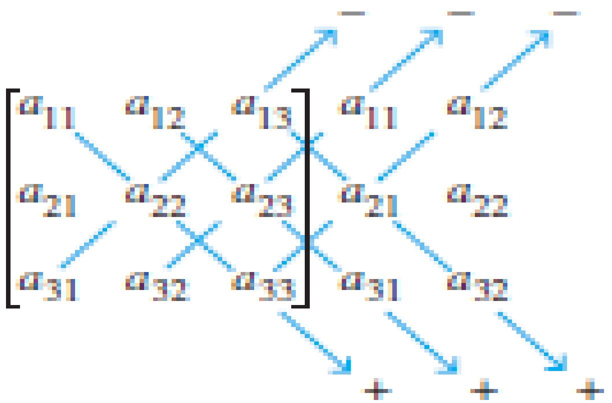
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

3. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \\ 3 & -4 & -6 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ -4 & -6 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 3 & -6 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}$, expanding along the first row

$$= 1(-6 + 12) - 3(-12 - 9) + 5(-8 - 3) = 6 + 63 - 55 = 14$$

To find the value of a 3x3 determinant, the following rule, called the Sarrus rule may also be useful.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$



Write the product of the elements of each leading diagonal with positive sign. Also write the product of the elements of each secondary diagonal with negative sign.

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \\ 3 & -4 & -6 \end{vmatrix} \qquad \begin{matrix} 1 & 3 & 5 & 1 & 3 \\ 2 & 1 & 3 & 2 & 1 \\ 3 & -4 & -6 & 3 & -4 \end{matrix}$$

$$1 \times 1 \times (-6) + 3 \times 3 \times 3 + 5 \times 2 \times (-4) - 3 \times 1 \times 5 - (-4) \times 3 \times 1 - (-6) \times 2 \times 3 = -6 + 27 - 40 - 15 + 12 + 36 = 75 - 61 = 14$$

Finding the determinant using row operations

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement} \\ -1 & \text{if } E \text{ is an interchange} \\ r & \text{if } E \text{ is a scale by } r \end{cases}$$

Find $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \\ 3 & -4 & -6 \end{vmatrix}$ using row operations

Finding the determinant using the cofactors

Definition If M_{ij} is the minor of an element a_{ij} of a determinant, then the cofactor of the element, denoted by A_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

Thus, the minor of a_{11} in D is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ and the cofactor is $A_{11} = (-1)^{1+1} M_{11} = M_{11}$.

Definition Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be a 3×3 determinant.

Then the 2×2 determinant obtained by deleting the row and column, in which an element a_{ij} lies, is called the **minor** of the element and is denoted by M_{ij}.

Write the cofactors of elements of second row of the determinant

$$D = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

and hence find the value of the determinant.

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -7 & 9 \end{vmatrix} = -(18 + 21) = -39$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix} = 9 - 6 = 3$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 2 & -7 \end{vmatrix} = -(-7 - 4) = 11$$

So

$$A_{21} = -39, \quad A_{22} = 3, \quad \text{and} \quad A_{23} = 11$$

Now

$$D = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = (-4)(-39) + 3(3) + 6(11) = 231$$

P1: If any two rows or columns of a determinant are identical, then its value equals zero.

Example

$$\begin{vmatrix} 2 & 3 & 1 \\ -3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 0 \text{ [The first and second rows are identical.]}$$

P2: If all the elements in any row or column of a determinant are zero, then the value of the determinant is also zero.

Example

$$\begin{vmatrix} 1 & -2 & 5 \\ 3 & 2 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0 \text{ [The elements of third row are all 0.]}$$

P4: The determinant of a diagonal matrix is equal to the product of the diagonal elements.

Example

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2 \times (-3) \times 4 = -24$$

If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

P5: The determinant of a square matrix equals the determinant of its transpose.

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 2 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 0 \\ 3 & 2 & -3 \end{vmatrix}$$

$\det A^T = \det A$.

P10: The determinant of a product of two square matrices is equal to the product of their determinants, i.e., $|AB| = |A||B|$.

EXAMPLE 5 Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

SOLUTION

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since $\det A = 9$ and $\det B = 5$,

$$(\det A)(\det B) = 9 \cdot 5 = 45 = \det AB \qquad \blacksquare$$

If any element of a row (or column) is the sum of two numbers then the determinant could be considered as the sum of other two determinants as follows

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + d_1 & b_2 + d_2 & b_3 + d_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Inverse of the matrix

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. In this case, C is an **inverse** of A. In fact, C is uniquely determined by A, because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix that is *not* invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

Formula The inverse of a nonsingular matrix A is given by the formula:

$$A^{-1} = \frac{1}{A}(\text{Adj.}A)$$

Note that no inverse of A exists, when $|A| = 0$.

Example

Find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Then the determinant of A is $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} =$ $0(2-3) - 1(1-9) + 2(1-6) = -2$. Since $|A| \neq 0$. $\therefore A^{-1}$ exists.

Now, the cofactors of the elements of A are

$$A_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \quad A_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5, \quad A_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6, \quad A_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -3$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1, \quad A_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$$

$$A_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$\therefore$ The matrix of cofactors is $\begin{bmatrix} -1 & 8 & 5 \\ 1 & -6 & 3 \\ 1 & 3 & -1 \end{bmatrix}$, hence

$$\text{Adj}(A) = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}.$$

Now

$$A^{-1} = \frac{1}{|A|}\text{Adj}(A) = \frac{1}{(-2)}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & -1 \end{bmatrix}$$

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

SOLUTION

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

It is a good idea to check the final answer:

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is not necessary to check that $A^{-1}A = I$ since A is invertible. ■

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

$$[\,A \quad I\,] \sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The $\det A \neq 0$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.