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## *Section 3*

- *solution set*
- *vector and matrix equations*
- *linear combinations, span and linear independent*

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## ➢ Solution set and parametric vector form

**EXAMPLE 3** Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$(0\ 0 \dots 0 \mid b \neq 0)$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \xrightarrow[r_3 \to r_3 + (-2) r_1]{r_2 \to r_2 + r_1} \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{pmatrix} \xrightarrow{r_3 \to r_3 + 3r_2} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$x_1, x_2$ Basic Var

$x_3$ Free Var.

$3x_2 = 6 \quad \therefore x_2 = 2, \quad 3x_1 + 5x_2 - 4x_3 = 7, \quad 3x_1 = 4x_3 - 3 \quad \therefore x_1 = \frac{4}{3}x_3 - 1 \ ; x_3 \in \mathbb{R}$

$$\rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{p} + \underbrace{x_3}_{t} \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{v}$$

$x_3 = t$

$x_2 = 2 \qquad \therefore t \in \mathbb{R}$

$x_1 = \frac{4}{3}t - 1$

$$\rightarrow \mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R})$$

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# ➢ Homogeneous system

**EXAMPLE 1** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$(0 \dots 0 \mid b \neq 0)$ $b = 0$

**SOLUTION** Let $A$ be the matrix of coefficients of the system and row reduce the augmented matrix $[\, A \quad \mathbf{0} \,]$ to echelon form:

pivot position

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow 3x_1 + 5x_2 - 4x_3 = 0$$

pivot col.

Solve for the basic variables $x_1$ and $x_2$ and obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with $x_3$ free. As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

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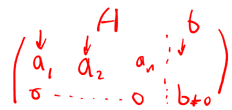
If every column of an augmented matrix contains a pivot, then the corresponding system is inconsistent




Whenever a system has free variables, the solution set contains many solutions.

augmented matrix [ $A$ **b** ] has a pivot position in every row, the corresponding system may or may not be consistent.



If the coefficient matrix $A$ has a pivot position in every row, then the corresponding system is consistent.

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## ➢ Matrix equation

$=$

$$\begin{aligned} x_1 + 2x_2 - \phantom{3}x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \qquad (1)$$

$$A = \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}_{2\times 3}, \quad b = \begin{pmatrix} 4 \\ 1 \end{pmatrix}_{2\times 1}$$

is equivalent to

$$\underbrace{\overset{A}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \overset{=}{=} \overset{b}{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}}_{} \qquad (3)$$

$$\mathbf{Ax = b} \leftarrow$$

$A =$ , $b =$

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# ➢ Vector equation

$$\begin{aligned} x_1 + 2x_2 - \phantom{3}x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \qquad (1)$$

is equivalent to

$b \; (x_1 \; x_2 \; x_3)$

$$\underset{(a_1}{x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \underset{a_2}{x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix}} + \underset{a_3}{x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix}} = \underset{b)}{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} \leftarrow \qquad (2)$$

If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \qquad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \qquad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{array}{ccccc} x_1 & x_2 & & x_n & \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array} \qquad (6)$$

$m$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$

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# ➢ Linear combination

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \dots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

V.eq

is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with **weights** $c_1, \dots, c_p$. ~~Property (ii)~~

$\underbrace{\begin{pmatrix}1\\0\\-1\end{pmatrix}}_{v_1}$ $\underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{v_2}$ $1v_1 + 1v_2 = \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{b_1}$, $1v_1 \ominus 1 v_2 = \underbrace{\begin{pmatrix}1\\-1\\-2\end{pmatrix}}_{b_2}$, $(-1)v_1 + (2)v_2 = \underbrace{\begin{pmatrix}-1\\2\\1\end{pmatrix}}_{b_3}$

R G B
0 → 255

$(v_1 \ v_2 \ \cdots \ v_p \mid y)$

Consistent y lin. com.

inconsistent y not lin. com.

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**EXAMPLE 5** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights $x_1$ and $x_2$ exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \qquad \cancel{(1)}$$

If vector equation (1) has a solution, find it.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \qquad \cancel{(3)}$$

To solve this system, row reduce the augmented matrix of the system as follows:[3]

$$\begin{array}{ccc} & & \\ \end{array}\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_1 \quad a_2 \quad b$

The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\underset{a_1}{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}} + 2\underset{a_2}{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}$$

■

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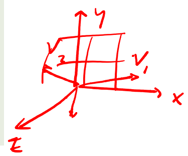
## ➢ span

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \ldots, \mathbf{v}_p$**. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.




R G B

$0 \to 255$

$\overset{x_1}{0}R + \overset{x_2}{0}G + \overset{x_3}{0}B =$ $0 \to 255$

{ set of all lin.con. } span

span{R, G,}B ⊆ ⊆ {R, G, B}

$y \in \text{span}\{v_1, v_2, \ldots, v_p\}$

y in the span { }

$(v_1 \ v_2 \ \ldots \ v_2 \ ; \ y)$

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**EXAMPLE 6** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then Span $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\mathbf{b}$ in that plane?

**SOLUTION** Does the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ have a solution? To answer this, row reduce the augmented matrix $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}\,]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$0 \quad 0 \quad \neq 0$

The third equation is $0 = -2$, which shows that the system has no solution. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has no solution, and so $\mathbf{b}$ is *not* in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$. ■

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The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of $A$.

Asking whether a vector $\mathbf{b}$ is in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\,\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_p \quad \mathbf{b}\,]$ has a solution.

In the next theorem, the sentence "The columns of $A$ span $\mathbb{R}^m$" means that *every* $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$. In general, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$—that is, if Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

$A$

every row in co. Matrix has pivot .

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Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^4$? Why or why not?

$v_1, v_2, \ldots, v_p \in \mathbb{R}^m$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ $4 \times 3$

max no pivot position $= \min\{\text{no. row}, \text{no. col.}\}$

$p < m$

not span $\mathbb{R}^m$

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$. Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$? Why or why not?

$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix}$

$\begin{bmatrix} -3 & 9 & -6 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $\begin{bmatrix} b \\ \\ \end{bmatrix}$

$0 \quad 0 \quad 0$ $\neq 0$

**EXAMPLE 3** Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $b_1, b_2, b_3$?

$\mathbb{R}^3$ span by col(s) of A? ✗ no

**SOLUTION** Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{array}\right]$$

$= 0$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

$b_1 - \frac{1}{2}b_2 + b_3 = 0$

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Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular $A$, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

c. The columns of $A$ span $\mathbb{R}^m$.

d. $A$ has a pivot position in every row.

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For what value(s) of $h$ will $\mathbf{y}$ be in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} \in \text{Span}$$

Lin. Com.

The vector $\mathbf{y}$ belongs to Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if there exist scalars $x_1, x_2, x_3$ such that

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \\ \quad v_1 \quad v_2 \quad v_3 \quad y \end{array} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & \underbrace{h-5}_{=0} \end{bmatrix} \neq \emptyset$$

$h - 5 = 0 \;\therefore\; h = 5$

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# Linear independent

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

$(v_1 \ v_2 \ - \ v_p \mid 0)$

$\overset{0}{x_1} v_1 + \overset{0}{x_2} v_2 + \overset{0}{x_3} v_3 = 0$ , $x_1 = x_2 = x_3 = 0$

$(v_1 \quad v_2 \quad v_3)$ lin. ind.

The columns of a matrix $A$ are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable. (trivial sol. → all var. are Basic)

**EXAMPLE 1** Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \quad \begin{matrix} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 3r_1 \end{matrix} \quad \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{pmatrix} \quad r_3 \to r_3 + 2r_2 \quad \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{cccc} & & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-3x_2 - 3x_3 = 0 \quad \therefore x_2 = -x_3$, $\quad x_1 + 4x_2 + 2x_3 = 0 \quad \therefore x_1 = 2x_3$

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ $\qquad 2x_3 v_1 - x_3 v_2 + x_3 v_3 = 0 \quad \to \quad x_3(2v_1 - v_2 + v_3) = 0$

$\therefore 2v_1 - v_2 + v_3 = 0 \qquad \therefore v_2 = 2v_1 + v_3$

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**EXAMPLE 2** Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

**SOLUTION** To study $A\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

At this point, it is clear that there are three basic variables and no free variables. So the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and the columns of $A$ are linearly independent. ■

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If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$< n$ not span

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

**EXAMPLE 6** Determine by inspection if the given set is linearly dependent.

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

lin.dep

b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

$v_1 \quad v_2 \quad v_3$

$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

dep

c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

$-\frac{3}{2}$

$\begin{pmatrix} -2 & 3 \\ 4 & -6 \\ 6 & -9 \\ 10 & 15 \end{pmatrix}$

$v_2 \to v_2 + 2v_1$

$v_3 \to v_3 + 3v_1$

$v_4 \to v_4 + 5v_1$

$\begin{pmatrix} -2 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 30 \end{pmatrix}$

$x_1 v_1 + x_2 v_2 + \cdots = 0$