Section 2
Solving system of linear equations

A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, $x_1, \ldots, x_n$. An example is

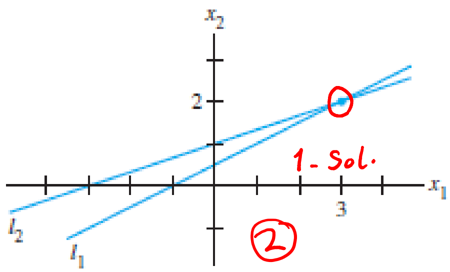
A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively. ~~For~~

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. ~~That is, each~~

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

$+$ $\qquad x_2 = 2 \qquad x_2 = 3$

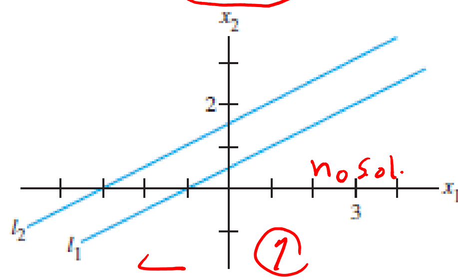
$(3, 2)$



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

$+$

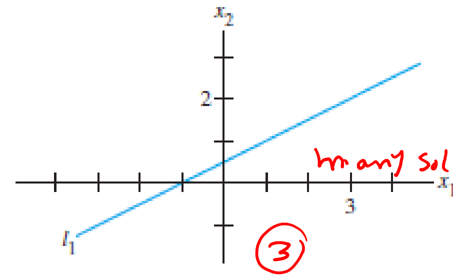
$0 = 2$



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

$-1\times$

$0 = 0$



A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Solve the following system

$$\begin{array}{rcl|l} x_1 - 2x_2 + x_3 &=& 0 & r_1 \\ 2x_2 - 8x_3 &=& 8 & r_2 \\ -4x_1 + 5x_2 + 9x_3 &=& -9 & r_3 \end{array}$$

$4r_1 + r_{3_{old}} \to r_{3_{new}}$

$$\begin{array}{rcl|l} x_1 - 2x_2 + x_3 &=& 0 & \\ 0x_1 + 2x_2 - 8x_3 &=& 8 & r_2 * \frac{1}{2} \\ 0x_1 - 3x_2 + 13x_3 &=& -9 & r_3 \end{array}$$

$\frac{1}{2}r_2 \to$

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 &=& 0 \\ 0x_1 + 1x_2 - 4x_3 &=& 4 \\ 0x_1 - 3x_2 + 13x_3 &=& -9 \end{array}$$

$r_{3_{new}} \to r_{3_{old}} + 3r_2$

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 &=& 0 \\ 0x_1 + 1x_2 - 4x_3 &=& 4 \\ 0x_1 + 0x_2 + 1x_3 &=& 3 \end{array}$$

$x_1 - 2x_2 + x_3 = 0 \quad \therefore x_1 =$

$x_2 - 4x_3 = 4$

$x_2 - 12 = 4 \quad \therefore x_2 = 16$

$x_3 = 3$

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[2] $r_{i_{new}} \to r_{i_{old}} + c * r_j$
2. (Interchange) Interchange two rows. $r_i \leftrightarrows r_j$
3. (Scaling) Multiply all entries in a row by a nonzero constant. $c * r_i$

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

non-zero row

zero row

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

leading

1st element in the row ≠ zero

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

reduced

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

pivot

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

echelon.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0\\ 2x_2 - 8x_3 &= 8\\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \cdots x_n \end{array} \\ \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \end{array} \qquad b = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

coefficient matrix

The augmented matrix is

$$(A : b) \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \quad r_3 \to r_3 + 4r_1 \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

= Free term

$$r_3 \to r_3 + \frac{3}{2} r_2 \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[\text{row op.}]{\frac{1}{2} r_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

pivot col.

pivot position

$x_3 = 3$

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Examples:

Determine if the following system is consistent:

$[0 \ \cdots \ 0 \ \vdots \ b]$ with b nonzero

$\rightarrow (A \vdots b)$, $b \neq 0$

in consistent has no sol.

$$
\begin{aligned}
x_2 - 4x_3 &= 8 \\
2x_1 - 3x_2 + 2x_3 &= 1 \\
5x_1 - 8x_2 + 7x_3 &= 1
\end{aligned}
$$

$$
\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \\
\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}
\end{array}
\quad r_2 \rightleftarrows r_1 \quad
\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}
$$

$$
r_3 \to r_3 + -\tfrac{5}{2} r_1 \quad
\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}
\quad r_3 \to r_3 + \tfrac{1}{2} r_2 \quad
\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}
$$

in Consistent

$0x_1 + 0x_2 + 0x_3 = 5/2$, $5/2 \neq 0$

$0 = \frac{5}{2}$

- Let $\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$ find the values of *h* and *k* such that The system :

a. has no solution.

[0 — 0 | b ≠ 0]

$$\begin{array}{cccc} x_1 & x_2 & & b \\ 2 & -1 & \vdots & h \\ -6 & 3 & \vdots & k \end{array}$$

$r_2 \to r_2 + 3r_1$

$$\begin{array}{cccc} x_1 & x_2 & & \\ 2 & -1 & \vdots & h \\ 0 & 0 & \vdots & k+3h \end{array}$$

≠ 0

a. $k + 3h \neq 0$

b. has ~~one~~ solution.

[ ]

all var. basic.

c. has many solution.

[ ]

There is a free var.

Free Var

$k + 3h = 0$ $k = -3h$

$$\begin{array}{ccc} x_1 & x_2 & \\ 2 & -1 & h \\ 0 & 0 & 0 \end{array}$$

- Find the solution of system of linear equations corresponding to the following augmented matrix.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\xrightarrow{r_1 \rightleftarrows r_4} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[r_3 \to r_3 + 2r_1]{r_2 \to r_2 + r_1} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[r_4 \to r_4 + \frac{3}{2} r_2]{r_3 \to r_3 + \frac{-5}{2} r_2} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Basic var

$$r_3 \rightleftarrows r_4 \sim \begin{array}{c} \begin{matrix} x_1 & x_2 & & x_4 & \end{matrix} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} & & x_3 & & \end{matrix} \end{array}$$

pivot position

pivot col. Free var.

$x_1 + 4x_2 + 5x_3 - 9\cancel{x_4} = -7$

$2x_2 + 4x_3 - 6\cancel{x_4} = -6$ $\quad \therefore x_2 = -2x_3 - 3$

$-5x_4 = 0$

$\therefore x_4 = 0$

$x_1 = 8x_3 + 12 - 5x_3 - 7$

$\therefore x_1 = 3x_3 + 5 \quad ; \; x_3 \in \mathbb{R}$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_3 + 5 \\ -2x_3 - 3 \\ x_3 \\ 0 \end{pmatrix}$$