

A **linear equation** in the variables x_1, \dots, x_n is an equation that can be written in the form

$$\sum a_i x_i = b \quad a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b \quad (1)$$

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, x_1, \dots, x_n . An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7 \end{aligned} \quad (2)$$

a_1, a_2, \dots, a_n
Coefficients

$$\{x_1, x_2, x_3\} = \{1, 5, 6\}$$

The set of all possible solutions is called the **solution set** of the linear system.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

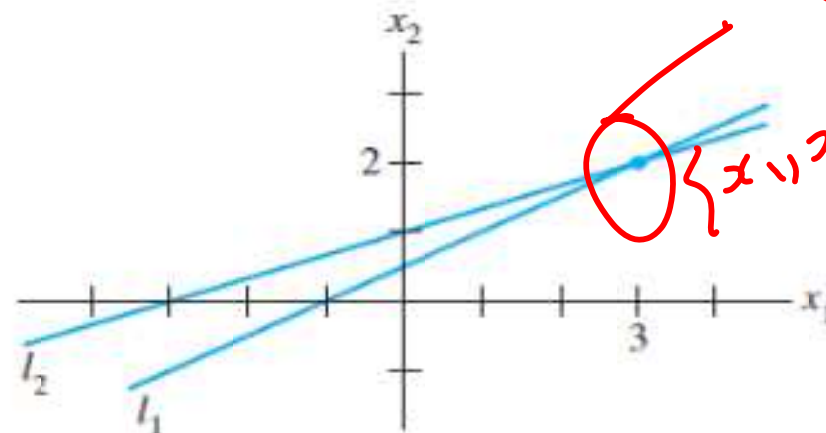
$$a_{11} = 1 \quad a_{12} = -2$$

$$a_{21} = -1 \quad a_{22} = 3$$

$$x_1 - 2x_2 = -1$$

$$-x_1 + 3x_2 = 3$$

The graphs of these equations are lines, which we denote by ℓ_1 and ℓ_2 . A pair of numbers (x_1, x_2) satisfies *both* equations in the system if and only if the point (x_1, x_2) lies on both ℓ_1 and ℓ_2 . In the system above, the solution is the single point $(3, 2)$, as you can easily verify. See Fig. 1.



Solution
one solution

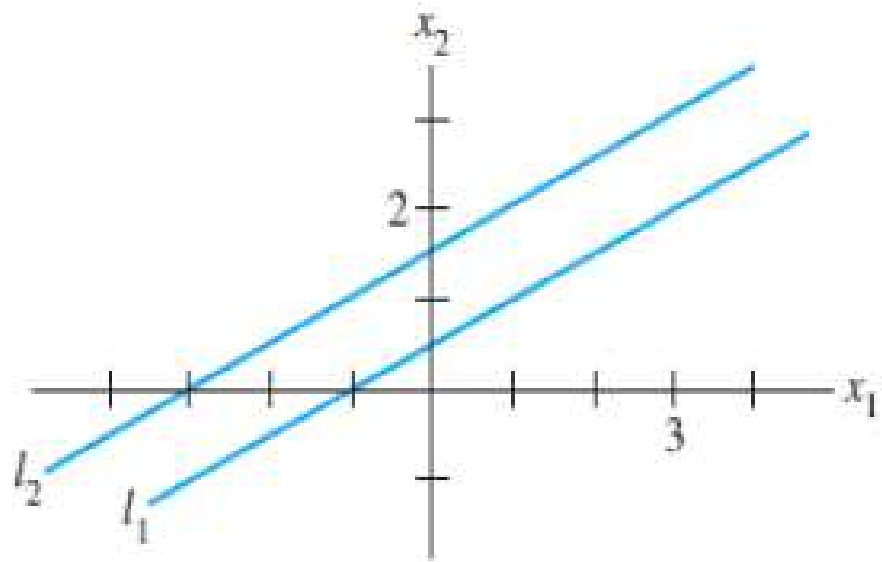
$\{x_1, x_2\}$

FIGURE 1 Exactly one solution.

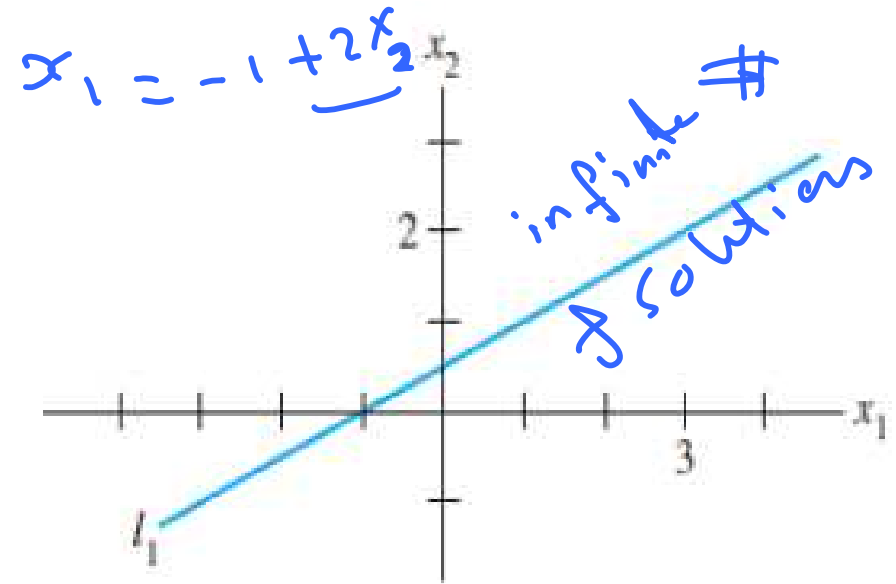
$$(a) \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

$$(b) \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

$x_1 - 2x_2 = -$



(a)



(b)

FIGURE 2 (a) No solution. (b) Infinitely many solutions.

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

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$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\-4x_1 + 5x_2 + 9x_3 &= -9\end{aligned}\tag{3}$$

*Coefficients
matrix*

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 2 & -8 & | & 8 \\ -4 & 5 & 9 & | & -9 \end{bmatrix}$$

*Augmented
matrix*

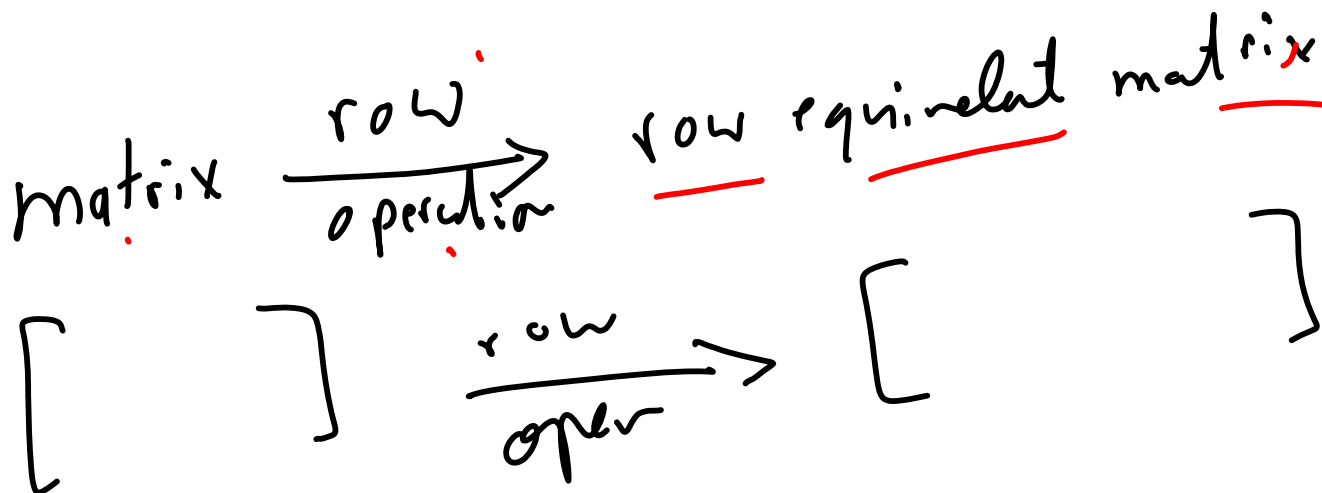
ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.²

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2. (Interchange) Interchange two rows.

3. (Scaling) Multiply all entries in a row by a nonzero constant.



If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

$$|1 - 3 \cdot (\frac{1}{3})| \neq 0 \quad \text{echlon}$$

$$\begin{array}{r} \text{row} \\ \text{op} \end{array} \Rightarrow \begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{array}$$

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{array}$$

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ 0 \quad x_2 - 4x_3 = 4 \\ 0 \quad 0 \quad x_3 = 3 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Augmented matrix

$$x_3 = 3 \rightarrow \begin{array}{l} x_2 - 4(3) = 4 \\ x_2 - 12 = 4 \end{array}$$

$$x_2 = 16$$

$$x_1 - 2(16) + 4 = 0$$

$$x_1 - 32 + 4 = 0$$

$$x_1 = 28$$

set of solⁿ = $\{28, 16, 3\}$ unique solⁿ

Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

inconsistent

row
operat

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

$$\frac{5}{2} - \frac{5}{5} = \frac{5}{2} - 1 = \frac{3}{2}$$

pract: 4

2, 3, 4, 1, 7

-2/5

$$\begin{bmatrix} 5 & -8 & 7 & 1 \\ 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -8 & 7 & 1 \end{bmatrix}$$

$$0x_1 + 0x_2 + 0x_3 = \frac{5}{2}$$

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

THEOREM 1

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

DEFINITION

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A . A **pivot column** is a column of A that contains a pivot position.

-11

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

not echelon form
not reduced.



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot

General form:

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns



$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

← Pivot column

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

New pivot column

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

← Row 1 + (-6) · row 3

← Row 2 + (-2) · row 3

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{2}$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row 1} + (9) \cdot \text{row 2}$$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{3}$$

Solutions of Linear Systems

$$\begin{array}{r} \rightarrow \\ x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{array}$$

Pivot
column

x_1, x_2 Basic variables
 x_3 free variable

$$x_1 - 5(x_3) = 1$$

$$x_2 = (x_3) + 4$$

$$0 = 0 \quad \checkmark$$

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$$x_5 = 7$$

$$x_4 =$$

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \cdots \ 0 \ b] \quad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & b \\ 0x_1 + 0x_2 + 0x_3 = 5 \end{array} \right]$$

Vectors in \mathbb{R}^2

A matrix with only one column is called a **column vector**, or simply a **vector**. Examples of vectors with two entries are

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} .2 \\ .3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$2 \times 1 \quad 2 \times 1 \quad 2 \times 1$

$$\text{if } \mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } c = 5, \quad \text{then } c\mathbf{u} = 5 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$$

The number c in $c\mathbf{u}$ is called a **scalar**;

Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$

$$4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}, \quad (-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$$

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

Geometric Descriptions of \mathbb{R}^2

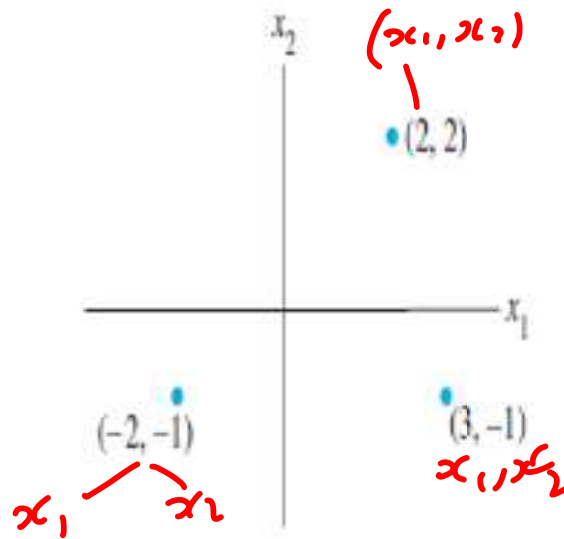


FIGURE 1 Vectors as points.

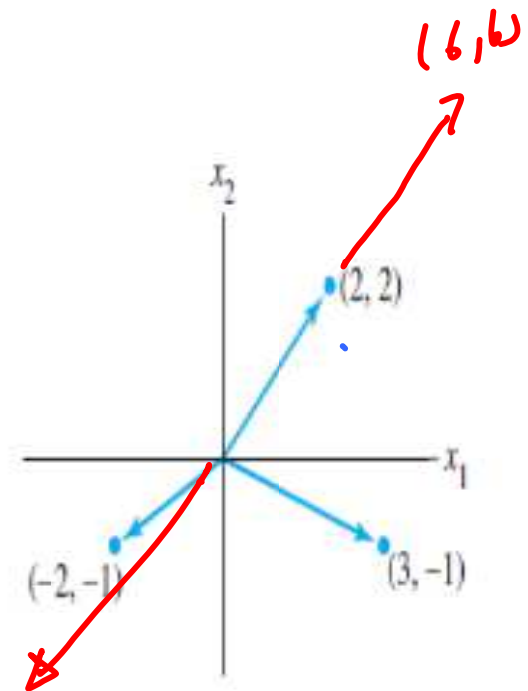


FIGURE 2 Vectors with arrows.

Parallelogram Rule for Addition

If \mathbf{u} and \mathbf{v} in \mathbb{R}^2 are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are \mathbf{u} , $\mathbf{0}$, and \mathbf{v} . See Fig. 3.

$\mathbf{0}$ = null vector

$$\mathbb{R}^2 \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbb{R}^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

$$\mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{0} \cdot \mathbf{u} = 0$$

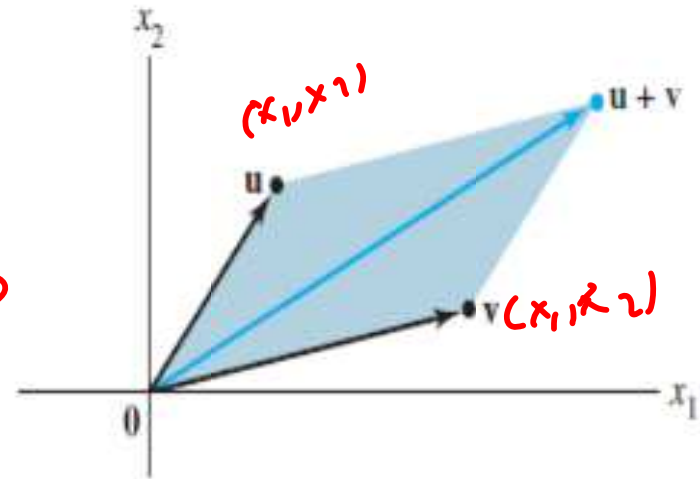


FIGURE 3 The parallelogram rule.

§ 2

EXAMPLE 2 The vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are displayed in Fig. 4. ■

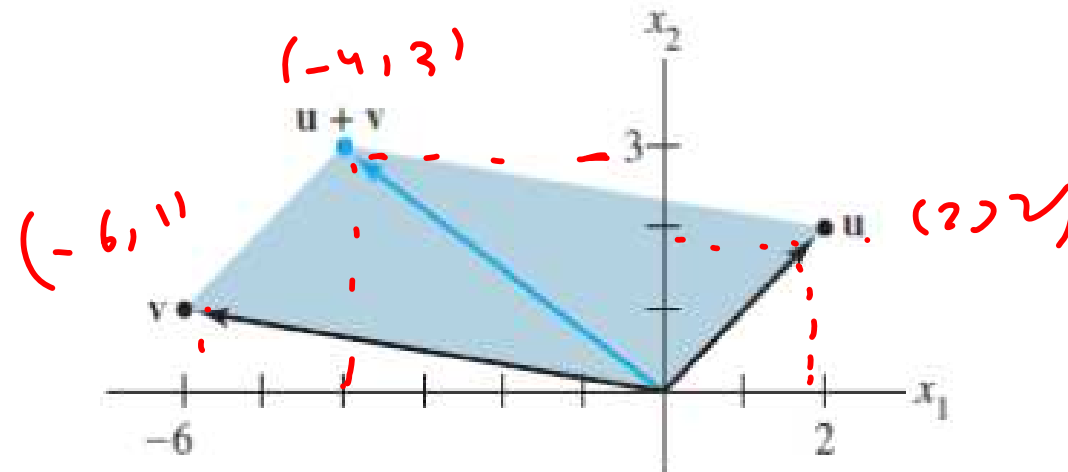
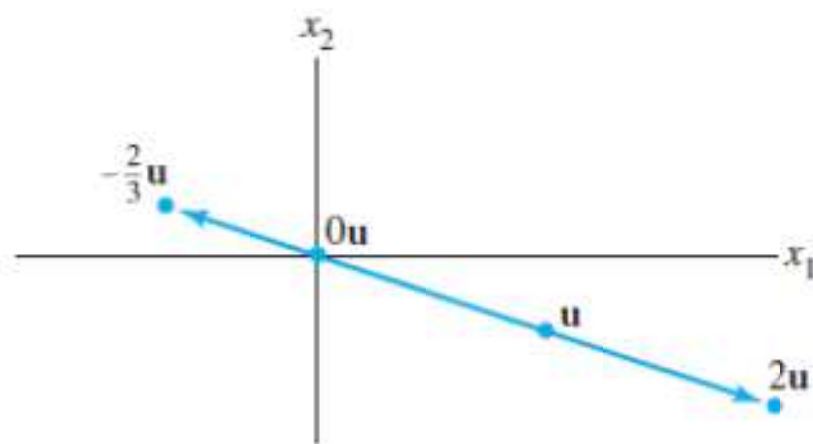


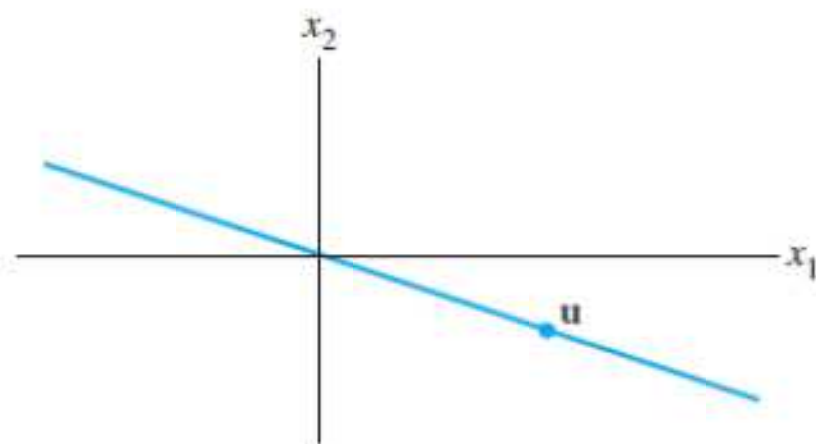
FIGURE 4

EXAMPLE 3 Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Display the vectors \mathbf{u} , $2\mathbf{u}$, and $-\frac{2}{3}\mathbf{u}$ on a graph.

SOLUTION See Fig. 5, where \mathbf{u} , $2\mathbf{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $-\frac{2}{3}\mathbf{u} = \begin{bmatrix} -2 \\ 2/3 \end{bmatrix}$ are displayed. The arrow for $2\mathbf{u}$ is twice as long as the arrow for \mathbf{u} , and the arrows point in the same direction. The arrow for $-\frac{2}{3}\mathbf{u}$ is two-thirds the length of the arrow for \mathbf{u} , and the arrows point in opposite directions. In general, the length of the arrow for $c\mathbf{u}$ is $|c|$ times the



Typical multiples of \mathbf{u}



The set of all multiples of \mathbf{u}

Vectors in \mathbb{R}^3

Vectors in \mathbb{R}^3

Vectors in \mathbb{R}^3 are 3×1 column matrices with three entries.

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}_{1 \times 3}$$

$$\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_{3 \times 1}$$

Vectors in \mathbb{R}^n

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Algebraic Properties of \mathbb{R}^n

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbb{R}^n and all scalars c and d :

- | | |
|---|--|
| (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ |
| (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ | (vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$ |
| (iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$ | (viii) $1\mathbf{u} = \mathbf{u}$ |

Vectors in \mathbb{R}^n

If n is a positive integer, \mathbb{R}^n (read “r-n”) denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

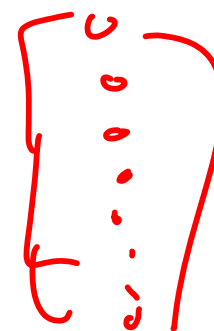
$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$. (The number of entries in $\mathbf{0}$ will be clear from the context.)

Algebraic Properties of \mathbb{R}^n

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbb{R}^n and all scalars c and d :

- | | |
|---|--|
| (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ |
| (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ | (vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$ |
| (iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
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Linear Combinations

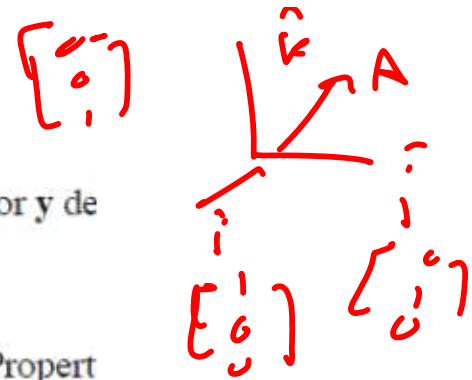
Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n and given scalars c_1, c_2, \dots, c_p , the vector \mathbf{y} defined by

$$\mathbf{y} = c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with **weights** c_1, \dots, c_p . Property above permits us to omit parentheses when forming such a linear combination.

c_j can be $\left\{ \begin{array}{l} \text{integer} \\ \text{fraction} \\ \text{complex} \end{array} \right.$

$$\mathbf{b} = c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2$$



$$\hat{\mathbf{A}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$$

EXAMPLE 5 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether \mathbf{b} can be generated (or written) as a linear combination of \mathbf{a}_1 and \mathbf{a}_2 . That is, determine whether weights x_1 and x_2 exist such that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 = \mathbf{b} \quad (1)$$

If vector equation (1) has a solution, find it.

SOLUTION Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

\uparrow \uparrow \uparrow
 \mathbf{a}_1 \mathbf{a}_2 \mathbf{b}

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence \mathbf{b} is a linear combination of \mathbf{a}_1 and \mathbf{a}_2 , with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$(3) \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + (2) \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$3\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{b}$$

$$x_1 = 3$$

$$x_2 = 2$$

■

A vector equation

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array} \right] \quad (5)$$

In particular, \mathbf{b} can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

2. The vector \mathbf{y} belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Any vector $\mathbf{y} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$

This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array} \right]$$

The system is consistent if and only if there is no pivot in the fourth column. That is, $h - 5$ must be 0. So \mathbf{y} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$.

Remember: The presence of a free variable in a system does not guarantee that the system is consistent.

25. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

- Is \mathbf{b} in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?
- Is \mathbf{b} in W ? How many vectors are in W ?
- Show that \mathbf{a}_1 is in W . [*Hint*: Row operations are unnecessary.]

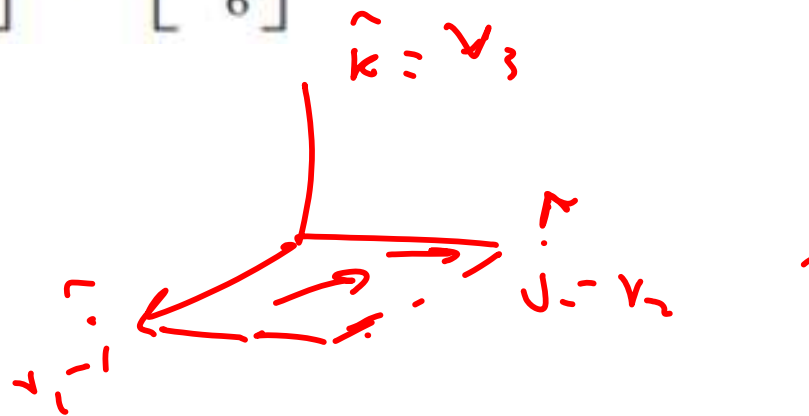
$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] \mathbf{b}$$

$$c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + c_3 \mathbf{a}_3 = \mathbf{b}$$

In Exercises 11 and 12, determine if \mathbf{b} is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 .

11. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

$\mathbb{R}^3 \{v_1, v_2, v_3\}$
 $\{v_1, v_2\}$



$$A = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$A = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$
$$= 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= 3v_1 + 2v_2 + 4v_3$$

$$v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$S = 3i + 2j$$

$$S = c_1 v_1 + c_2 v_2$$

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in \mathbb{R}^n , then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of \mathbb{R}^n spanned (or generated) by $\mathbf{v}_1, \dots, \mathbf{v}_p$** . That is, $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

with c_1, \dots, c_p scalars.

Asking whether a vector \mathbf{b} is in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\mathbf{v}_1 \ \dots \ \mathbf{v}_p \ \mathbf{b}]$ has a solution.

A Geometric Description of $\text{Span}\{\mathbf{v}\}$ and $\text{Span}\{\mathbf{u}, \mathbf{v}\}$

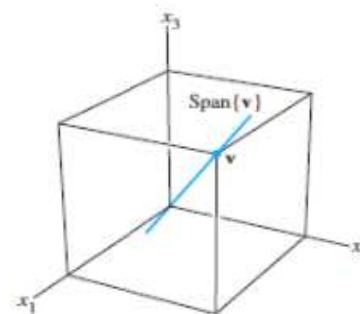


FIGURE 10 $\text{Span}\{\mathbf{v}\}$ as a line through the origin.

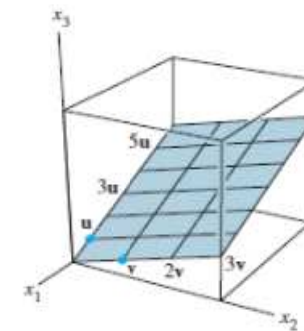


FIGURE 11 $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

EXAMPLE 6 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then

$\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane through the origin in \mathbb{R}^3 . Is \mathbf{b} in that plane?

SOLUTION Does the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ have a solution? To answer this, row reduce the augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{b}]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The third equation is $0 = -2$, which shows that the system has no solution. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has no solution, and so \mathbf{b} is *not* in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$. ■

In Exercises 13 and 14, determine if \mathbf{b} is a linear combination of the vectors formed from the columns of the matrix A .

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

15. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$. For what value(s) of h is \mathbf{b} in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 ?

THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{x} is in \mathbb{R}^n , then the product of A and \mathbf{x} , denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in \mathbf{x} as weights; that is,

$$A\mathbf{x} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

EXAMPLE 1

$$\begin{aligned} \text{a. } \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} &= 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{aligned}$$

$$\text{b. } \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix} \quad \blacksquare$$

EXAMPLE 2 For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in \mathbb{R}^m , write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

SOLUTION Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5 , and 7 into a vector \mathbf{x} . That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x} \quad \blacksquare$$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{b} is in \mathbb{R}^m , the matrix equation

$$A\mathbf{x} = \mathbf{b} \quad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \quad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}] \quad (6)$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A .

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3 ?

SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array} \right] &\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{array} \right] \end{aligned}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every \mathbf{b} because some choices of \mathbf{b} can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

$$b_1 - \frac{1}{2}b_2 + b_3 = 0$$

If A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

1.5 SOLUTION SETS OF LINEAR SYSTEMS

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in \mathbb{R}^m . Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in \mathbb{R}^n). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector \mathbf{x} that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

$$\begin{aligned}
 3x_1 + 5x_2 - 4x_3 &= 0 \\
 -3x_1 - 2x_2 + 4x_3 &= 0 \\
 6x_1 + x_2 - 8x_3 &= 0
 \end{aligned}$$

Solve

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 - \frac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3 \mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

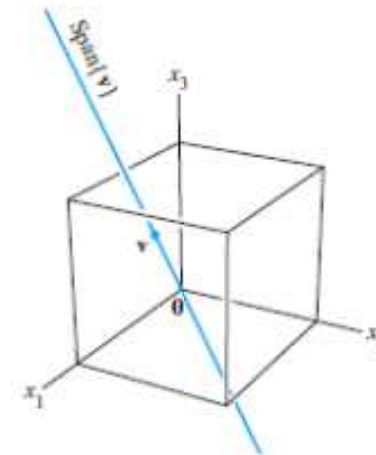


FIGURE 1

13)

$$10x_1 - 3x_2 - 2x_3 = 0$$

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix} \\ &= x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} \quad (\text{with } x_2, x_3 \text{ free}) \\ &\quad \uparrow \quad \quad \quad \uparrow \\ &\quad \mathbf{u} \quad \quad \quad \mathbf{v} \end{aligned}$$

EXAMPLE 2 A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous “system”

$$10x_1 - 3x_2 - 2x_3 = 0 \quad (1)$$

SOLUTION There is no need for matrix notation. Solve for the basic variable x_1 in terms of the free variables. The general solution is $x_1 = .3x_2 + .2x_3$, with x_2 and x_3 free. As a vector, the general solution is

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix} \\ &= x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} \quad (\text{with } x_2, x_3 \text{ free}) \quad (2) \end{aligned}$$

$\begin{matrix} \uparrow \\ \mathbf{u} \end{matrix}$
 $\begin{matrix} \uparrow \\ \mathbf{v} \end{matrix}$

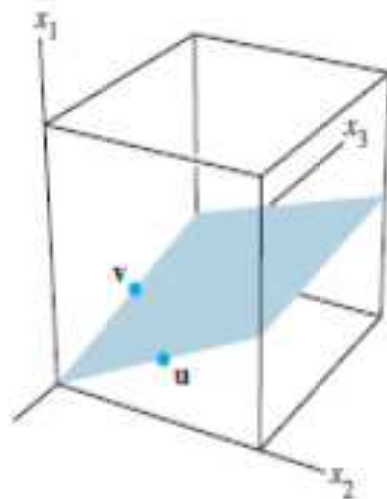


FIGURE 2

This calculation shows that every solution of (1) is a linear combination of the vectors \mathbf{u} and \mathbf{v} , shown in (2). That is, the solution set is $\text{Span}\{\mathbf{u}, \mathbf{v}\}$. Since neither \mathbf{u} nor \mathbf{v} is a scalar multiple of the other, the solution set is a plane through the origin. See Fig. 2. ■

EXAMPLE 3 Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

SOLUTION Here A is the matrix of coefficients from Example 1. Row operations on $[A \ \mathbf{b}]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{l} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \\ 0 = 0 \end{array}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free. As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

\uparrow \mathbf{p} \uparrow \mathbf{v}

1.7 LINEAR INDEPENDENCE

DEFINITION

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights c_1, \dots, c_p , not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

linear depend if not all $c_i \neq 0$

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

- Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
- If possible, find a linear dependence relation among \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 .

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array} \qquad \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \qquad \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free. Choose any nonzero value for x_3 —say, $x_3 = 5$. Then $x_1 = 10$ and $x_2 = -5$. Substitute these values into equation (1) and obtain

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

This is one (out of infinitely many) possible linear dependence relations among \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 . ■

Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

EXAMPLE 2 Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

SOLUTION To study $A\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

At this point, it is clear that there are three basic variables and no free variables. So the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and the columns of A are linearly independent. ■

Sets of One or Two Vectors

A set containing only one vector—say, \mathbf{v} —is linearly independent if and only if \mathbf{v} is not the zero vector. This is because the vector equation $x_1 \mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$. The zero vector is linearly dependent because $x_1 \mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

The next example will explain the nature of a linearly dependent set of two vectors.

EXAMPLE 3 Determine if the following sets of vectors are linearly independent.

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

SOLUTION

- a. Notice that \mathbf{v}_2 is a multiple of \mathbf{v}_1 , namely, $\mathbf{v}_2 = 2\mathbf{v}_1$. Hence $-2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, which shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.
- b. The vectors \mathbf{v}_1 and \mathbf{v}_2 are certainly *not* multiples of one another. Could they be linearly dependent? Suppose c and d satisfy

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}$$

If $c \neq 0$, then we can solve for \mathbf{v}_1 in terms of \mathbf{v}_2 , namely, $\mathbf{v}_1 = (-d/c)\mathbf{v}_2$. This result is impossible because \mathbf{v}_1 is *not* a multiple of \mathbf{v}_2 . So c must be zero. Similarly, d must also be zero. Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set. ■

1.5 SOLUTION SETS OF LINEAR SYSTEMS

The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

EXAMPLE 1 Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 - \frac{4}{3}x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{array}$$

Solve for the basic variables x_1 and x_2 and obtain $x_1 = \frac{4}{3}x_3, x_2 = 0$, with x_3 free.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3 \mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

vector in $x_1 - x_3$ plane

\therefore span of all vector $x_3 \mathbf{v}$

Ex

$$10x_1 - 3x_2 - 2x_3 = 0$$

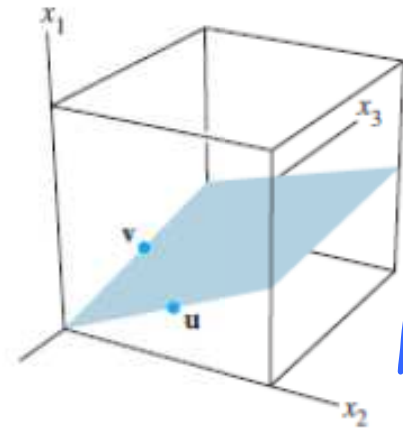
$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix} \\ &= x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} \quad (\text{with } x_2, x_3 \text{ free}) \end{aligned}$$

$$u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

u, v
Linearly independent

The solution form plane



Parametric Vector Form

parametric vector equation

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \text{ in } \mathbb{R})$$

Solutions of Nonhomogeneous Systems

EXAMPLE 3 Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{l} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \\ 0 = 0 \end{array}$$

$x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

\uparrow \mathbf{p} \uparrow \mathbf{v}

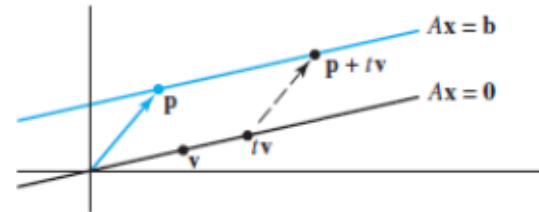
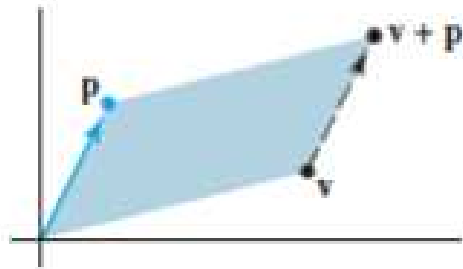
$$A\mathbf{x} = \mathbf{0} \quad \mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R})$$

t: free parameter

[with the same \mathbf{v} that appears in (3)]. Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector \mathbf{p} to the solutions of $A\mathbf{x} = \mathbf{0}$. The vector \mathbf{p} itself is just one particular

The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter, solution of $A\mathbf{x} = \mathbf{b}$ [corresponding to $t = 0$ in (3)]. ■

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R})$$



Suppose the equation $Ax = \mathbf{b}$ is consistent for some given \mathbf{b} , and let \mathbf{p} be a solution. Then the solution set of $Ax = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where \mathbf{v}_h is any solution of the homogeneous equation $Ax = \mathbf{0}$.

Warning: Theorem 6 and Fig. 6 apply only to an equation $Ax = \mathbf{b}$ that has at least one nonzero solution \mathbf{p} . When $Ax = \mathbf{b}$ has no solution, the solution set is empty.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution \mathbf{x} as a vector whose entries depend on the free variables, if any.
4. Decompose \mathbf{x} into a linear combination of vectors (with numeric entries) using the free variables as parameters.

1. Each of the following equations determines a plane in \mathbb{R}^3 . Do the two planes intersect? If so, describe their intersection.

$$\begin{aligned}x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9\end{aligned}$$

2. Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form, and relate the solution set to the one found in Example 2.

$$\begin{aligned}\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} &\sim \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix} \\ x_1 + 3x_3 &= 4 \\ x_2 - 2x_3 &= -1\end{aligned}$$

$$\{x_1 = 4 - 3x_3, x_2 = -1 + 2x_3, \text{ with } x_3 \text{ free.}\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

\uparrow \uparrow
 \mathbf{p} \mathbf{v}

The augmented matrix $[10 \ -3 \ -2 \ 7]$ is row equivalent to $[1 \ -3 \ -2 \ .7]$, and the general solution is $x_1 = .7 + .3x_2 + .2x_3$, with x_2 and x_3 free. That is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 + .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} \\ = \mathbf{p} + x_2 \mathbf{u} + x_3 \mathbf{v}$$

The solution set of the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is the translated plane $\mathbf{p} + \text{Span}\{\mathbf{u}, \mathbf{v}\}$, which passes through \mathbf{p} and is parallel to the solution set of the homogeneous equation in Example 2.

LINEAR SYSTEMS